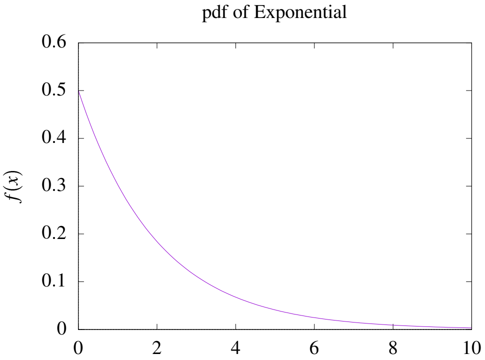
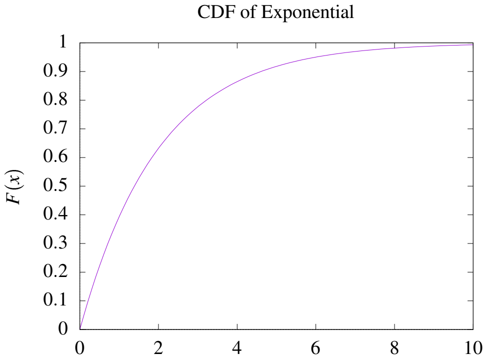
$Exponential(\mu)$ and $Geometric(p)$

October 2, 2025

The $Exponential(\mu)$ Distribution and Variate

What $f(x)$ models random arrivals?





- If there are n arrivals in the time interval $[0, a_n]$, then the average interarrival time is $\bar{r} = a_n/n$
- The mean of the $f(x)$ is $\bar{r}$, by **convention** the variable μ is used in the exponential pdf. In the case of SSQs: $\mu = \bar{r}$.
- The variance of $f(x)$ is $\sigma^2 = \mu^2$.
- $f(x)$ crosses the y-axis at $1/\mu$

Here we are talking about a probability distribution again — what do you think is next?

Variate for $Exponential(\mu)$

Warning: *The textbook uses* ***average interarrival time*** $\bar{r} = \mu = \frac{a_n}{n} = \frac{1}{\lambda}$, *where* λ *is the* ***arrival rate*** *(jobs/sec). About half the statistics and simulation textbooks use* λ, *the other half use* $\mu = \frac{a_n}{n}$. *Neither is right or wrong — just be careful when blindly following recipes from other sources.*

The exponential pdf is

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

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The exponential pdf is

$$f(x) = \frac{1}{\mu} \mathrm{e}^{-\frac{x}{\mu}}$$

and the first step is integration to find the cdf

$$F(x) = \int_0^x \frac{1}{\mu} \mathrm{e}^{-\frac{t}{\mu}} \, \mathrm{d}t = -\mathrm{e}^{-\frac{t}{\mu}} \Big|_0^x = -\mathrm{e}^{-\frac{x}{\mu}} - (-1) = 1 - \mathrm{e}^{-\frac{x}{\mu}}$$

Variate for $Exponential(\mu)$

The second step is setting $F(x) = u$ (which we determine programmatically with `Random()`) and solving for x

$$
\begin{aligned}
u &= 1 - e^{-\frac{x}{\mu}} \\
e^{-\frac{x}{\mu}} &= 1 - u \\
-\frac{x}{\mu} &= \ln(1 - u) \\
x &= -\mu \ln(1 - u)
\end{aligned}
$$

Since u is random in $(0, 1)$, $1 - u$ is random in $(0, 1)$ as well and without loss of generality, we can write:

$$x = -\mu \ln(1 - u) = -\mu \ln(u) \qquad \mu = \frac{a_n}{n} = \frac{1}{\lambda}$$

See why we prefer the range of $Random()$ to be $(0, 1)$?

λ is the **arrival rate** (jobs/sec)

$Geometric(p)$ the discrete analogue to $Exponential(\mu)$

We have seen that $Uniform(a,b) \in \Re$ has a discrete analogue of $Equilikely(a,b)$; it is fair to ask the same about $Exponential(\mu)$.
Let

$$y = \lfloor Exponential(\mu) \rfloor$$

then what is the discrete probability distribution of y?

$$Pr(y \geq 1) = Pr(Exponential(\mu) \geq 1) = Pr(-\mu \ln(u) \geq 1) \tag{1}$$

Well this isn't that hard, since u is from `Random()` it has a simple distribution and we have hope...

$$\begin{aligned} -\mu \ln(u) &\geq 1 \\ \ln(u) &\leq -\frac{1}{\mu} \\ u &\leq e^{-\frac{1}{\mu}} \end{aligned}$$

$Geometric(p)$ the discrete analogue to $Exponential(\mu)$

Conclusion:

$$\lfloor Exponential(\mu) \rfloor \geq 1 \quad \text{when } u \text{ (from } \texttt{Random()}\text{) is} \quad \leq e^{-\frac{1}{\mu}}$$

But we would like a simpler way to think of $e^{-\frac{1}{\mu}}$

When $\mu > 0$, we have

$$0 < e^{-\frac{1}{\mu}} < 1$$

and we think of it as a probability

$$p = e^{-\frac{1}{\mu}} \tag{2}$$

You may be familiar with the geometric distribution reflecting the number of “heads” flipped before the first tail with a p-bias coin.

A Variate for $Geometric(p)$

Now consolidating the algebra from (eqn 1)...

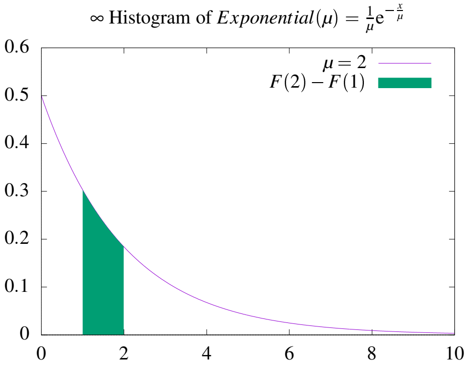
$$
\begin{aligned}
y &= \lfloor Exponential(\mu) \rfloor = \lfloor -\mu \ln(u) \rfloor = \left\lfloor \frac{\ln(u)}{-\frac{1}{\mu}} \right\rfloor \\
y &= \left\lfloor \frac{\ln(u)}{\ln\left(e^{-\frac{1}{\mu}}\right)} \right\rfloor
\end{aligned}
$$

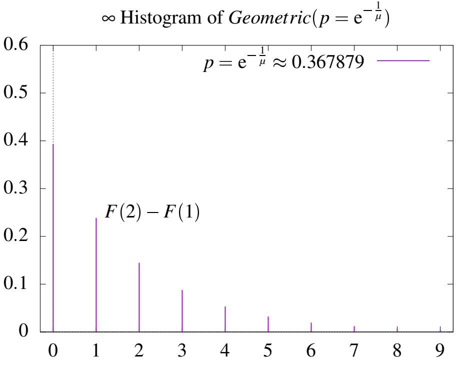
...and plugging in our defintion of p in terms of μ (eqn 2):

$$
y = \left\lfloor \frac{\ln(u)}{\ln(p)} \right\rfloor = \left\lfloor \frac{\ln(1-u)}{\ln(p)} \right\rfloor
$$

With u from `Random()`, y will follow the geometric distribution parameterized with probability p.

The $Exponential(\mu)$ and $Geometric(p = e^{-\frac{1}{\mu}})$ Variate distributions





- Exponential distribution: pdf $f(\mu,x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}$, $F(\mu,x) = 1 - e^{-\frac{x}{\mu}}$
- Geometric distribution: **point mass function** $g(\mu,x) = F(\mu,x+1) - F(\mu,x)$
- Exponential variate: $x = -\mu\ln(u) \equiv -\mu\ln(1-u)$
- Geometric variate: $y = \left\lfloor \frac{\ln u}{\ln p} \right\rfloor \equiv \left\lfloor \frac{\ln(1-u)}{\ln p} \right\rfloor$

Thought experiment: how is an SSQ's job arrival rate of $\lambda \approx \frac{e}{4}$ connected to flipping a fair coin?