

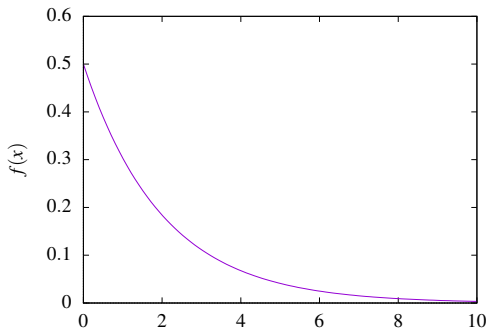
Exponential(μ) and Geometric(p)

September 22, 2026

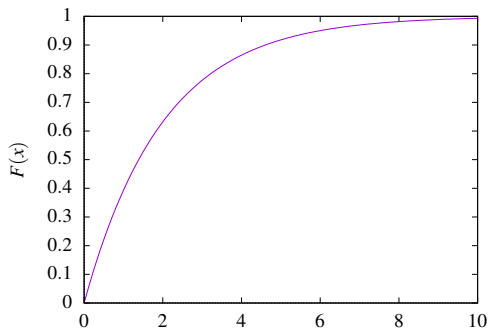
The $\text{Exponential}(\mu)$ Distribution and Variate

What $f(x)$ models random arrivals?

pdf of Exponential



CDF of Exponential



- ▶ If there are n arrivals in the time interval $[0, a_n]$, then the average interarrival time is $\bar{r} = a_n/n$
- ▶ The mean of the $f(x)$ is $\bar{r}$, by **convention** the variable μ is used in the exponential pdf. In the case of SSQs: $\mu = \bar{r}$.
- ▶ The variance of $f(x)$ is $\sigma^2 = \mu^2$.
- ▶ $f(x)$ crosses the y-axis at $1/\mu$

Here we are talking about a probability distribution again — what do you think is next?

Variate for *Exponential*(μ)

Warning: The textbook uses **average interarrival time** $\bar{r} = \mu = \frac{a_n}{n} = \frac{1}{\lambda}$, where λ is the **arrival rate** (jobs/sec). About half the statistics and simulation textbooks use λ , the other half use $\mu = \frac{a_n}{n}$. Neither is right or wrong — just be careful when blindly following recipes from other sources.

The exponential pdf is

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

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The exponential pdf is

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and the first step is integration to find the cdf

$$F(x) = \int_0^x \frac{1}{\mu} e^{-\frac{t}{\mu}} dt = -e^{-\frac{t}{\mu}} \Big|_0^x = -e^{-\frac{x}{\mu}} - (-1) = 1 - e^{-\frac{x}{\mu}}$$

Variate for *Exponential*(μ)

The second step is setting $F(x) = u$ (which we determine programmatically with `Random()`) and solving for x

$$\begin{aligned}u &= 1 - e^{-\frac{x}{\mu}} \\e^{-\frac{x}{\mu}} &= 1 - u \\-\frac{x}{\mu} &= \ln(1 - u) \\x &= -\mu \ln(1 - u)\end{aligned}$$

Since u is random in $(0, 1)$, $1 - u$ is random in $(0, 1)$ as well and without loss of generality, we can write:

$$x = -\mu \ln(1 - u) = -\mu \ln(u) \qquad \mu = \frac{a_n}{n} = \frac{1}{\lambda}$$

See why we prefer the range of *Random()* to be $(0, 1)$?

λ is the **arrival rate** (jobs/sec)

Geometric(p) the discrete analogue to *Exponential*(μ)

We have seen that $Uniform(a, b) \in \mathfrak{R}$ has a discrete analogue of *Equilikely*(a, b); it is fair to ask the same about *Exponential*(μ).

Let

$$y = \lfloor Exponential(\mu) \rfloor$$

then what is the discrete probability distribution of y ?

$$Pr(y \geq 1) = Pr(Exponential(\mu) \geq 1) = Pr(-\mu \ln(u) \geq 1) \quad (1)$$

Well this isn't that hard, since u is from `Random()` it has a simple distribution and we have hope...

$$\begin{aligned} -\mu \ln(u) &\geq 1 \\ \ln(u) &\leq -\frac{1}{\mu} \\ u &\leq e^{-\frac{1}{\mu}} \end{aligned}$$

Geometric(p) the discrete analogue to *Exponential*(μ)

Conclusion:

$$\lfloor \text{Exponential}(\mu) \rfloor \geq 1 \quad \text{when } u \text{ (from Random()) is } \leq e^{-\frac{1}{\mu}}$$

But we would like a simpler way to think of $e^{-\frac{1}{\mu}}$

When $\mu > 0$, we have

$$0 < e^{-\frac{1}{\mu}} < 1$$

and we think of it as a probability

$$p = e^{-\frac{1}{\mu}} \tag{2}$$

You may be familiar with the geometric distribution reflecting the number of times a particular “side” of a p -bias coin will be flipped, before the “other side” occurs.

A Variate for *Geometric*(p)

Now consolidating the algebra from (eqn 1)...

$$y = \lfloor \text{Exponential}(\mu) \rfloor = \lfloor -\mu \ln(u) \rfloor = \left\lfloor \frac{\ln(u)}{-\frac{1}{\mu}} \right\rfloor$$
$$y = \left\lfloor \frac{\ln(u)}{\ln\left(e^{-\frac{1}{\mu}}\right)} \right\rfloor$$

... and plugging in our definition of p in terms of μ (eqn 2):

$$y = \left\lfloor \frac{\ln(u)}{\ln(p)} \right\rfloor = \left\lfloor \frac{\ln(1-u)}{\ln(p)} \right\rfloor \quad (3)$$

With u from `Random()`, y will follow the geometric distribution parameterized with probability p .

A Variate for *Geometric*(p)

$$y = \left\lfloor \frac{\ln(u)}{\ln(p)} \right\rfloor = \left\lfloor \frac{\ln(1-u)}{\ln(p)} \right\rfloor \quad (3)$$

IMPORTANT: In **simulation** we are typically interested in **number of successes (*heads*) before the first failure (*tails*)** (think MTBF). The variate above models this when p is the probability of success (heads).

$$y \in \{0, 1, 2, 3, 4, \dots\}$$

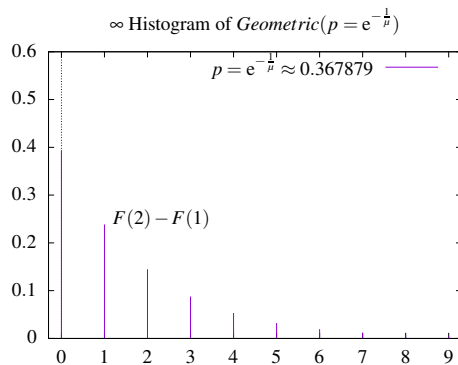
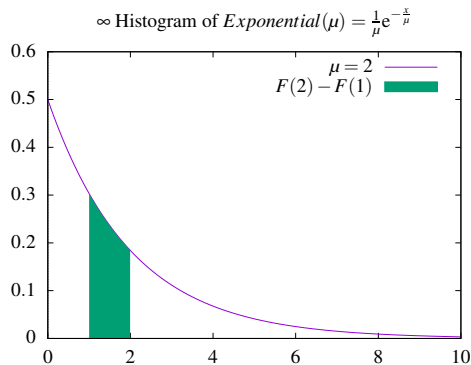
You may be more familiar with the geometric distribution being the total number of flips till the first failure (tail); in which case the outcomes are clearly in the set $\{1, 2, 3, 4, 5, \dots\}$.

... or the number of failures (probability $1 - p$) before the first success,

... or the number of flips till the first success.

With a little thought and the correct value for p , the variate above can be used for all of these. The text, slides and programming projects will **always** use eqn (3) for *Geometric*(p) with its corresponding interpretation..

The $Exponential(\mu)$ and $Geometric(p = e^{-\frac{1}{\mu}})$ Variate distributions



- ▶ Exponential distribution: pdf $f(\mu, x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}$, $F(\mu, x) = 1 - e^{-\frac{x}{\mu}}$
- ▶ Geometric distribution: **point mass function** $g(\mu, x) = F(\mu, x+1) - F(\mu, x)$
- ▶ Exponential variate: $x = -\mu \ln(u) \equiv -\mu \ln(1-u)$
- ▶ Geometric variate: $y = \left\lfloor \frac{\ln u}{\ln p} \right\rfloor \equiv \left\lfloor \frac{\ln(1-u)}{\ln p} \right\rfloor$

Thought experiment: how is an SSQ's job arrival rate of $\lambda \approx \frac{6}{4}$ connected to flipping a fair coin?