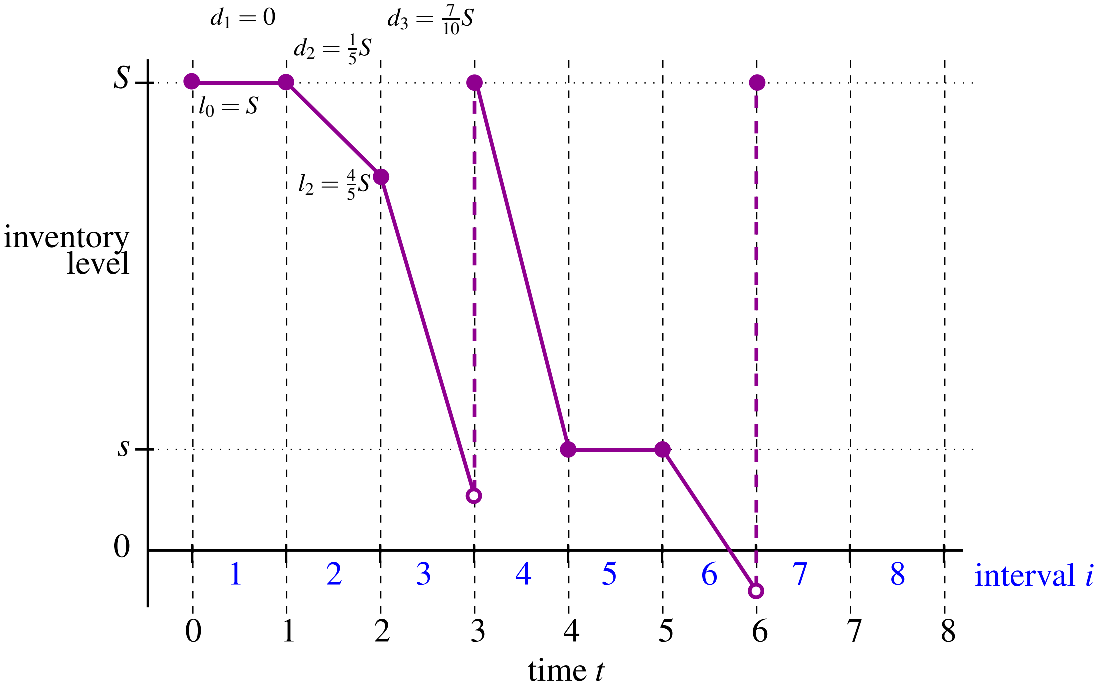
Section 1.3: A Simple Inventory System

- Distributes items from current inventory to customers
- Customer demand is discrete
- Simple $\Longleftrightarrow$ one type of item

Inventory Policy

- *Transaction Reporting*
  - Inventory review after *each* transaction
  - Significant labor may be required
  - Less likely to experience shortage
- *Periodic Inventory Review*
  - Inventory review is periodic
  - Items are ordered, if necessary, only at review times
  - (s, S) are the min,max inventory levels, $0 \leq s < S$
- We assume periodic inventory review
- Search for (s, S) that minimize cost

- time $t \in [0, n]$ (ordinal) intervals $i = 1, 2, 3, \ldots, n$
  integral values of t is when the $i = t$ interval **ends** and the $i = t + 1$ interval **begins**
- Sim state ($\mathcal{S}$) **changes** at the end of intervals, so at $t = 1, 2, \ldots$.
- l_{i-1}: inventory at $t = i - 1$, when the $i - 1$ interval ends and the i interval begins, $l_0 = S$
- o_{i-1}: inventory ordered at $t = i - 1$, $o_0 = 0$
- d_i: customer demand **during** the i-th interval
- Inventory **before** ordering may be negative.



$d_1 = 0$
$d_2 = \frac{1}{5}S$
$d_3 = \frac{7}{10}S$
S
$l_0 = S$
$l_2 = \frac{4}{5}S$
inventory level
s
0
1 2 3 4 5 6 7 8
interval i
0 1 2 3 4 5 6 7 8
time t

- Inventory level reviewed at the end of intervals

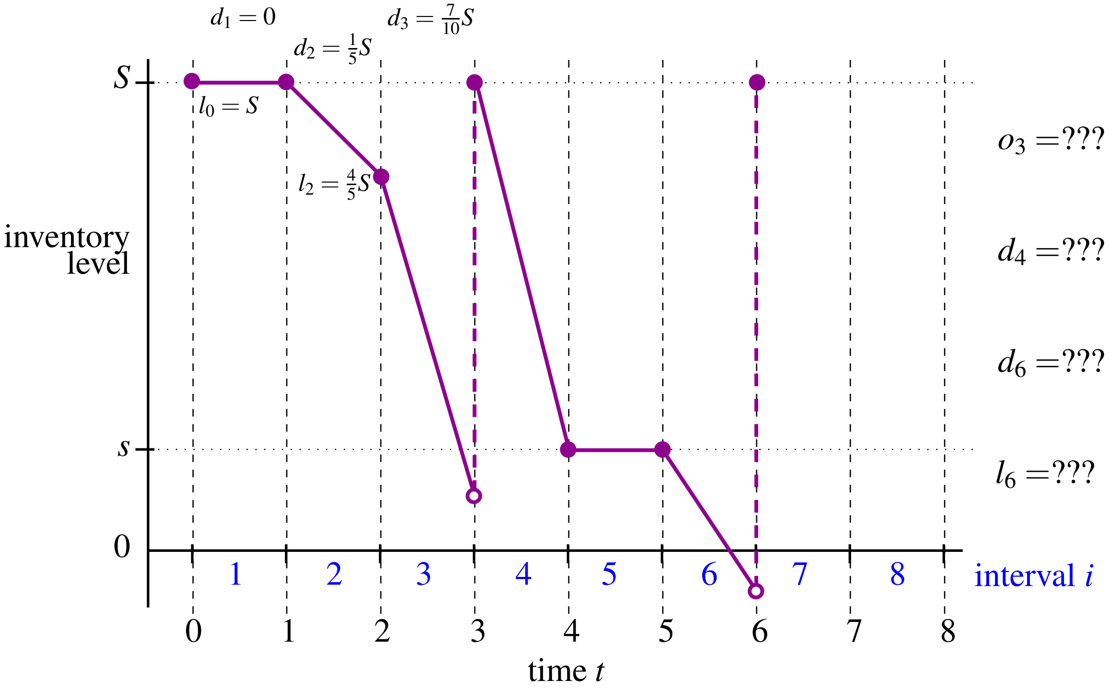
$$t = 1, 2, 3, \ldots, i \quad i = 1, 2, 3, \ldots$$

$$o_t = \begin{cases} 0 & l_t \geq s \\ S - l_t & l_t < s \end{cases} \qquad \underbrace{o_{i-1} = \begin{cases} 0 & l_{i-1} \geq s \\ S - l_{i-1} & l_{i-1} < s \end{cases}}_{\text{author's notation}}$$

- inventory delivered **immediately**

- inventory at the end of i-th interval l_i $(= l_{t+1})$

$$l_i = l_{i-1} - d_i + o_{i-1}$$



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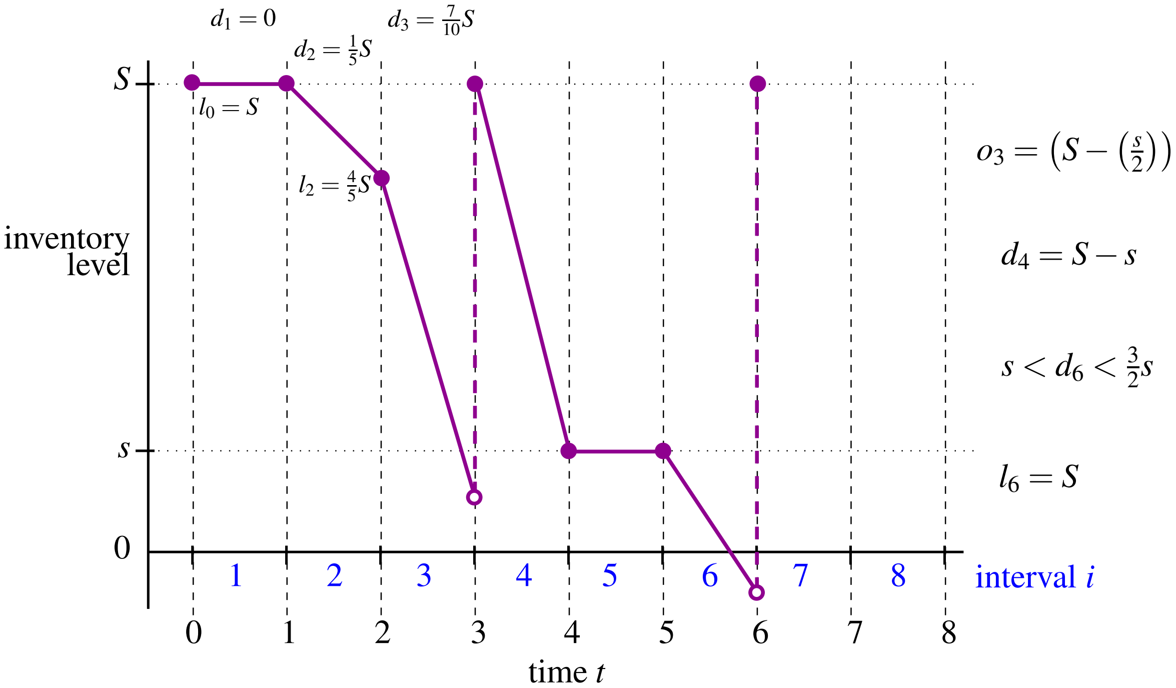
S
$l_0 = S$
$l_2 = \frac{4}{5}S$
inventory level
s
0
$o_3 =???$
$d_4 =???$
$d_6 =???$
$l_6 =???$
1 2 3 4 5 6 7 8
interval i
0 1 2 3 4 5 6 7 8
time t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$d_3 = \frac{7}{10}S$
S
$l_0 = S$
$l_2 = \frac{4}{5}S$
inventory level
s
0
$o_3 = \left(S - \left(\frac{s}{2}\right)\right)$
$d_4 = S - s$
$s < d_6 < \frac{3}{2}s$
$l_6 = S$
1 2 3 4 5 6 7 8
interval i
0 1 2 3 4 5 6 7 8
time t

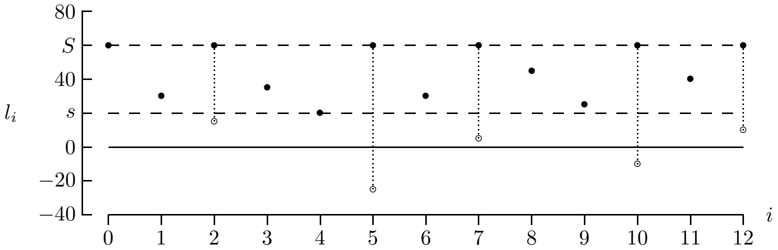
Time Evolution of Inventory Level

Algorithm 1.3.1

l_0 = S;   /* the initial inventory level is S */
i = 0;
while (*more demand to process*) {
    i++;
    if ($l_{i-1} < s$)
        o_{i-1} = S - l_{i-1};
    else
        o_{i-1} = 0;
    d_i = GetDemand();
    l_i = l_{i-1} + o_{i-1} - d_i;
}
n = i;
o_n = S - l_n;
l_n = S;   /* the terminal inventory level is S */
return $l_1, l_2, \ldots, l_n$ and $o_1, o_2, \ldots, o_n$;

Let $(s, S) = (20, 60)$ and consider $n = 12$ time intervals

i	:	1	2	3	4	5	6	7	8	9	10	11	12
d_i	:	30	15	25	15	45	30	25	15	20	35	20	30



Output Statistics

- What statistics to compute?
- *Average demand* and *average order*

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i \qquad\qquad \bar{o} = \frac{1}{n}\sum_{i=1}^{n} o_i.$$

- For Example 1.3.1 data

$$\bar{d} = \bar{o} = 305/12 \simeq 25.42 \text{ items per time interval}$$

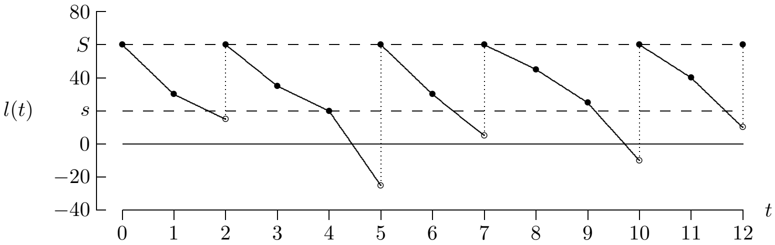
Flow Balance

- Average demand and order *must* be equal
- Ending inventory level is S
- Over the simulated period, all demand is satisfied
- Average “flow” of items in equals average “flow” of items out

- The inventory system is *flow balanced*

Constant Demand Rate

- Holding and shortage costs are proportional to time-averaged inventory levels
- Must know inventory level for all t
- *Assume* the demand rate is constant between review times

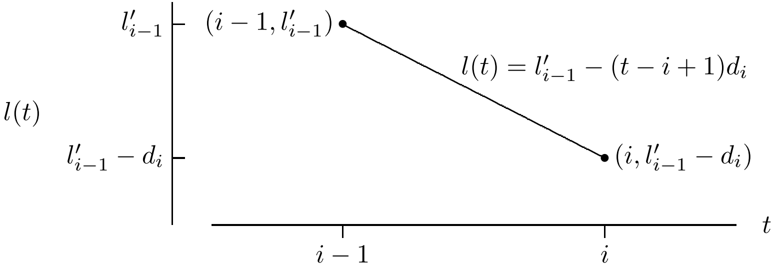


Inventory Level as a Function of Time

- The inventory level at any time t in i^{th} interval is

$$l(t) = l'_{i-1} - (t - i + 1)d_i$$

  if demand rate is constant between review times



- $l'_{i-1} = l_{i-1} + o_{i-1}$ represents inventory level *after* review

Time-Averaged Inventory Level

- $l(t)$ is the basis for computing the time-averaged inventory level
- Case 1: If $l(t)$ remains non-negative over i^{th} interval

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)dt$$

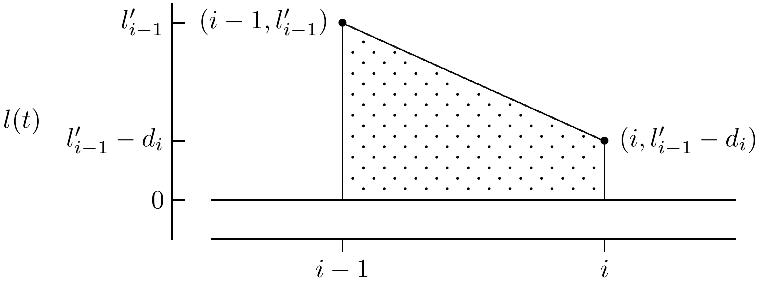
- Case 2: If $l(t)$ becomes negative at some time τ

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)dt \qquad \bar{l}_i^- = -\int_{\tau}^{i} l(t)dt$$

- $\bar{l}_i^+$ is the time-averaged *holding level*
  $\bar{l}_i^-$ is the time-averaged *shortage level*

Case 1: No Back Ordering

- No shortage during i^{th} time interval iff. $d_i \le l'_{i-1}$

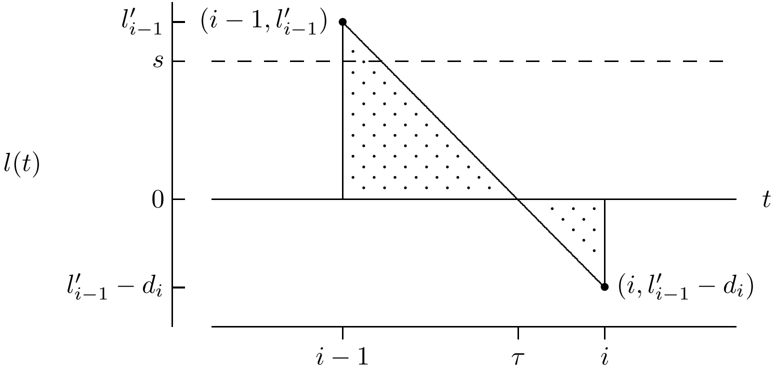


- Time-averaged holding level: area of a trapezoid

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)\,dt = \frac{l'_{i-1} + (l'_{i-1} - d_i)}{2} = l'_{i-1} - \frac{1}{2} d_i$$

Case 2: Back Ordering

- Inventory becomes negative iff. $d_i > l'_{i-1}$



Case 2: Back Ordering (Cont.)

- $l(t)$ becomes negative at time $t = \tau = i - 1 + (l'_{i-1}/d_i)$
- Time-averaged holding and shortage levels for i^{th} interval computed as the areas of triangles

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)\,dt = \cdots = \frac{(l'_{i-1})^2}{2d_i}$$

$$\bar{l}_i^- = -\int_{\tau}^{i} l(t)\,dt = \cdots = \frac{(d_i - l'_{i-1})^2}{2d_i}$$

Time-Averaged Inventory Level

- *Time-averaged holding level* and *time-averaged shortage level*

$$\bar{l}^{+} = \frac{1}{n}\sum_{i=1}^{n} \bar{l}_i^{+} \qquad\qquad \bar{l}^{-} = \frac{1}{n}\sum_{i=1}^{n} \bar{l}_i^{-}$$

- Note that time-averaged shortage level is positive
- The *time-averaged inventory level* is

$$\bar{l} = \frac{1}{n}\int_0^n l(t)\,dt = \bar{l}^{+} - \bar{l}^{-}$$

Conceptual Model

Inventory System Costs

- *Holding cost:* *for items in inventory*
- *Shortage cost:* *for unmet demand*
- *Setup cost:* *fixed cost when order is placed*
- *Item cost:* *per-item order cost*
- *Ordering cost:* *sum of setup and item costs*

Additional Assumptions

- *Back ordering is possible*
- *No delivery lag*
- *Initial inventory level is S*
- *Terminal inventory level is S*

Operating Costs

A facility's cost of operation is determined by:

- c_{item} : *unit cost of new item*
- c_{setup} : *fixed cost for placing an order*
- c_{hold} : *cost to hold one item for one time interval*
- c_{short} : *cost of being short one item for one time interval*

Case Study

- Automobile dealership that uses weekly periodic inventory review
- The facility is the showroom and surrounding areas
- The items are new cars
- The supplier is the car manufacturer
- "...customers are people convinced by clever advertising that their lives will be improved significantly if they purchase a new car from this dealer." (S. Park)
- Simple (one type of car) inventory system

If all parameters remain the same, **except** for s (the threshold inventory value **under which** we order cars for maximum inventory, S)...

How will the total dependent cost components change if s is **increased**?

- **Setup cost** (overhead for each order)
- **Holding cost** (cost to keep an unsold car on the lot)
- **Shortage cost** (cost of allowing customers to back-order cars)

Per-Interval Average Operating Costs

- The average operating costs *per time interval* are

  - *item cost :* $c_{\text{item}} \cdot \bar{o}$
  - *setup cost :* $c_{\text{setup}} \cdot \bar{u}$
  - *holding cost :* $c_{\text{hold}} \cdot \bar{l}^{+}$
  - *shortage cost :* $c_{\text{short}} \cdot \bar{l}^{-}$

- The average *total* operating cost *per time interval* is their sum
- For the stats and costs of the hypothetical dealership:

  - *item cost :* $\$8000 \cdot 29.29 = \$234,320$ *per week*
  - *setup cost :* $\$1000 \cdot 0.39 = \$390$ *per week*
  - *holding cost :* $\$25 \cdot 42.40 = \$1,060$ *per week*
  - *shortage cost :* $\$700 \cdot 0.25 = \$175$ *per week*

Cost Minimization

- By varying s (and possibly S), an optimal policy can be determined
- Optimal $\iff$ minimum average cost
- Note that $\bar{o} = \bar{d}$, and $\bar{d}$ depends only on the demands
- Hence, item cost is independent of (s, S)
- Average *dependent* cost is

$$\text{avg setup cost} + \text{avg holding cost} + \text{avg shortage cost}$$

Case Study Values

- Limited to a maximum of $S = 80$ cars
- Inventory reviewed every Monday (once a week)
- If inventory falls below $s = 20$ order cars sufficient to restore inventory to S
- (For now) no delivery lag
- Costs

$$
\begin{aligned}
c_{item} &= \$8000 && \text{per item} \\
c_{setup} &= \$1000 && \text{per order} \\
c_{hold} &= \$25 && \text{per car per week} \\
c_{short} &= \$700 && \text{per car per week}
\end{aligned}
$$

Example 1.3.7: Simulation Results