Traditional One Pass s^2 (in slides)

$$s^2 = \frac{1}{n}\Sigma\left(x_i - \bar{x}\right)^2 \tag{1}$$
$$= \frac{1}{n}\left\{\Sigma x_i^2 - \Sigma 2x_i\bar{x} + \bar{x}^2\right\} \tag{2}$$
$$= \frac{1}{n}\Sigma x_i^2 - 2\bar{x}\left(\frac{1}{n}\Sigma x_i\right) + \frac{1}{n}\Sigma\bar{x}^2 \tag{3}$$
$$= \frac{1}{n}\Sigma x_i^2 - 2\bar{x}^2 + \bar{x}^2 \tag{4}$$
$$= \frac{1}{n}\Sigma x_i^2 - \bar{x}^2 \tag{5}$$

Welford One Pass $\bar{x}_i$ and v_i

Definitions to begin with (**not the Welford equations**):

$$\bar{x}_i = \frac{1}{i}\left(x_1 + x_2 + \cdots + x_i\right) \tag{6}$$
$$Var_i = \left(\frac{1}{i}\Sigma x_i^2\right) - \bar{x}_i^2 \tag{7}$$
$$v_i = i \cdot Var_i = \left(x_1^2 + x_2^2 + \cdots + x_i^2\right) - i\bar{x}_i^2 \tag{8}$$

Equation (6) is the arithmetic mean of sample $\mathcal{X} = \{x_1, x_2, \ldots, x_i\}$, equation (7) is the (algebraically correct but numerically challenged) one-pass variance, and equation (8) is $\mathcal{X}$'s **variance** multiplied by i. Equation (8) is simply the **expanded notation** of one pass difference of squares (5).

Welford One Pass $\bar{x}_i$ Derivation

Begin with eqn 6...

$$\bar{x}_i = \frac{1}{i}\left(x_1 + x_2 + \cdots + x_i\right)$$

...and recognize $\bar{x}_{i-1}$ inside of it...

$$i\bar{x}_i = \underbrace{x_1 + x_2 + \cdots + x_{i-1}}_{(i-1)\bar{x}_{i-1}} + x_i$$
$$i\bar{x}_i = (i-1)\bar{x}_{i-1} + x_i$$
$$i\bar{x}_i = i\bar{x}_{i-1} - \bar{x}_{i-1} + x_i$$

...collect terms without the i factor:

$$i\bar{x}_i = i\bar{x}_{i-1} + \left(x_i - \bar{x}_{i-1}\right)$$

...solve for $\bar{x}_i$

$$\bar{x}_i = \bar{x}_{i-1} + \frac{1}{i}\left(x_i - \bar{x}_{i-1}\right) \tag{9}$$

Observe that with $x_0 \equiv 0$ equation 9 at x_1 is $\bar{x}_1 = 0 + \frac{1}{1}(x_1 - 0) = x_1$.

Welford One Pass v_i Derivation

Begin with equation 8...

$$v_i = i \cdot Var_i = x_1^2 + x_2^2 + \cdots + x_i^2 - i\bar{x}_i^2$$

$$v_i = \underbrace{x_1^2 + x_2^2 + \cdots + x_{i-1}^2}_{\text{will become } v_{i-1}\ldots} + x_i^2 - i\bar{x}_i^2 \tag{10}$$

and (like Welford $\bar{x}_i$ derivation) think of $x_1^2 + x_2^2 + \cdots + x_{i-1}^2$ in terms of v_{i-1}:

$$v_{i-1} = x_1^2 + x_2^2 + \cdots + x_{i-2}^2 + x_{i-1}^2 - \underbrace{(i-1)\bar{x}_{i-1}^2}_{\text{move to lhs}}$$

$$v_{i-1} + (i-1)\bar{x}_{i-1}^2 = x_1^2 + x_2^2 + \cdots + x_{i-1}^2 \tag{11}$$

$$v_i = \underbrace{x_1^2 + x_2^2 + \cdots + x_{i-1}^2}_{\text{substitute (11) into (10) for } x_1^2 \text{ through } x_{i-1}^2} + x_i^2 - i\bar{x}_i^2$$

$$v_i = v_{i-1} + (i-1)\bar{x}_{i-1}^2 + x_i^2 - i\bar{x}_i^2$$

distribute...

$$v_i = v_{i-1} + i\bar{x}_{i-1}^2 - \bar{x}_{i-1}^2 + x_i^2 - i\bar{x}_i^2$$

factor i out of terms ...

$$v_i = v_{i-1} - i\left(\bar{x}_i^2 - \bar{x}_{i-1}^2\right) + \left(x_i^2 - \bar{x}_{i-1}^2\right)$$

we have two separate differences of squares, factor! ...

$$v_i = v_{i-1} - i\underbrace{(\bar{x}_i - \bar{x}_{i-1})}_{\text{ugh}}(\bar{x}_i + \bar{x}_{i-1}) + (x_i - \bar{x}_{i-1})(x_i + \bar{x}_{i-1}) \tag{12}$$

Equation (12) would be factorable if we had x_i instead of $\bar{x}_i$. Let's fix that now: recall Welford equation 9, solved for $(\bar{x}_i - \bar{x}_{i-1})$:

$$\bar{x}_i = \bar{x}_{i-1} + \frac{1}{i}(x_i - \bar{x}_{i-1})$$

$$(\bar{x}_i - \bar{x}_{i-1}) = \frac{1}{i}(x_i - \bar{x}_{i-1}) \tag{13}$$

Substitute (13) into (12), the $\frac{i}{i}$ cancel ...

$$v_i = v_{i-1} - i\left\{\frac{1}{i}(x_i - \bar{x}_{i-1})\right\}(\bar{x}_i + \bar{x}_{i-1}) + (x_i - \bar{x}_{i-1})(x_i + \bar{x}_{i-1}) \tag{14}$$

Why did we do this? To get two $(x_i - \bar{x}_{x-1})$ for factoring! Rearranging terms to the right of v_{i-1} in positive first order...

$$v_i = v_{i-1} + (x_i - \bar{x}_{i-1})(x_i + \bar{x}_{i-1}) - (x_i - \bar{x}_{i-1})(\bar{x}_i + \bar{x}_{i-1})$$

$$v_i = v_{i-1} + (x_i - \bar{x}_{i-1})\{(x_i + \bar{x}_{i-1}) - (\bar{x}_i + \bar{x}_{i-1})\}$$

$$v_i = v_{i-1} + (x_i - \bar{x}_{i-1})(x_i - \bar{x}_i)$$

Aside from x_i, we want all the RHS terms to be from the previous iteration $(i-1)$, so use Welford's equation (9) to replace $\bar{x}_i$ ...

$$v_i = v_{i-1} + (x_i - \bar{x}_{i-1}) \left(x_i - \left\{ \bar{x}_{i-1} + \frac{1}{i}(x_i - \bar{x}_{i-1}) \right\} \right)$$

$$v_i = v_{i-1} + (x_i - \bar{x}_{i-1}) \left\{ (x_i - \bar{x}_{i-1}) - \frac{1}{i}(x_i - \bar{x}_{i-1}) \right\}$$

Notice the pattern $\alpha = x_i - \bar{x}_{i-1}$ in the RHS expression ...

$$\alpha \left(\alpha - \frac{\alpha}{i} \right) = \alpha^2 \left(\frac{i-1}{i} \right)$$

and we have Welford's equation for $v_i = i \cdot Var_i$:

$$v_i = v_{i-1} + \frac{i-1}{i}(x_i - \bar{x}_{i-1})^2 \tag{15}$$

Observe that (with $x_0 \equiv 0$ and $v_0 \equiv 0$) at $i = 1$ (the first datapoint) $v_1 = 0 + \frac{1-1}{1}(x_1 - 0)^2 = 0$.