Stationary Arrival Processes

- Generated using $Exponential(\mu)$ interarrival times
- $\mu = \frac{1}{\lambda}$ where $\lambda = \frac{n}{t}$, average arrivals per time unit
- Also called a homogeneous arrival process
- The **next arrival equation** is

$$a_{i+1} = a_i + Exponential\left(\frac{1}{\lambda}\right)$$

Generate n Arrival Times

we don't really do this in a simulation!

$a_0 \leftarrow 0$
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repeat `n times (`
  $a_{i+1} \leftarrow a_i + Exponential(\frac{1}{\lambda})$
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return $a_1, a_2, \ldots, a_n$

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A convenient fiction, arrival processes are often **time** dependent: $\lambda(t)$.

Non-Stationary Arrival Processes

Let the average arrival rate vary with t, you are given $\lambda(t)$.

Convert this stationary arrival equation to a non-stationary one:

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(where t is the **simulation clock**)

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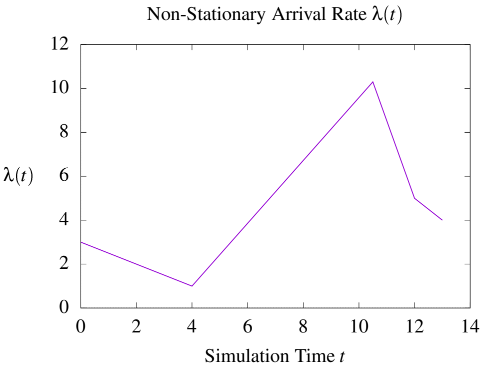
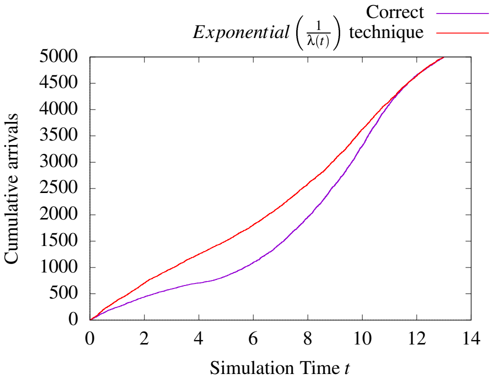
(where t is the **simulation clock**)

But does this technique actually work?

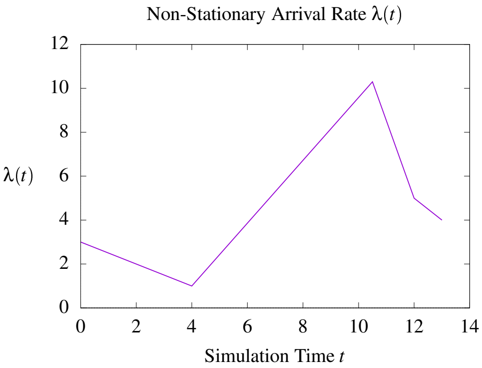
Compare Cumulative Arrivals between two techniques

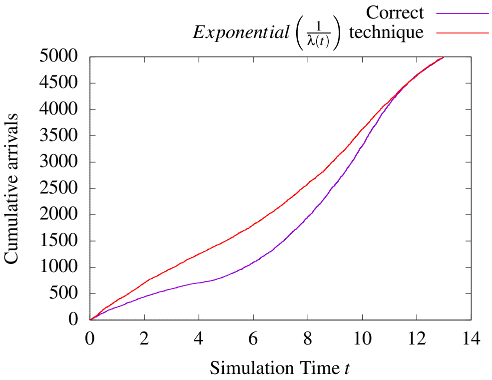




Clearly, there is evidence it doesn't work... You'll develop your own evidence in a future (next?) LGA.

Compare Cumulative Arrivals between two techniques

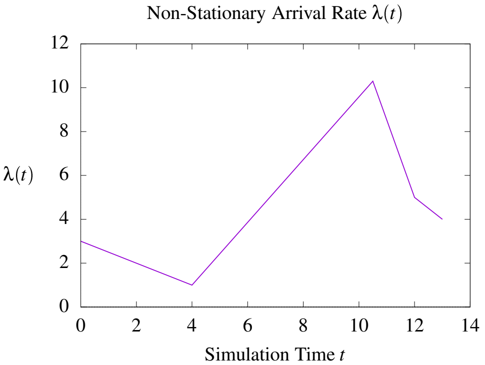


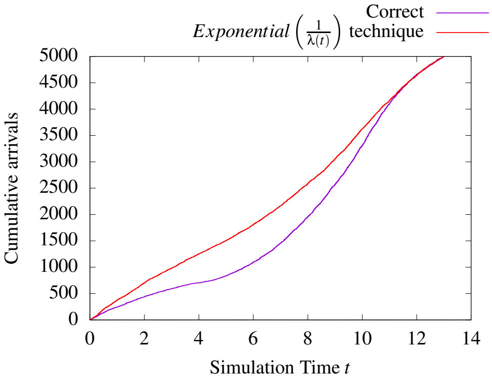


Clearly, there is evidence it doesn't work. . . You'll develop your own evidence in a future (next?) LGA.

Take 60s and discuss in your group — about how many simulations were required to generate the cumulative arrival plot on the right? All the information is on this screen. . .

Compare Cumulative Arrivals between two techniques



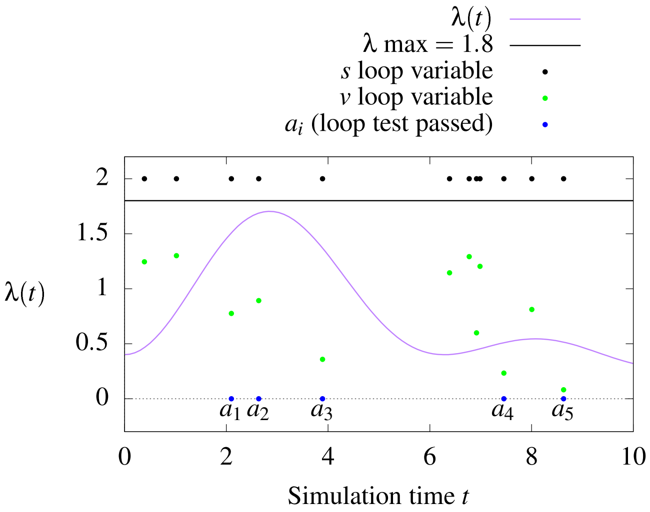


The left hand plot shows a **rate vs time relationship** — so the expected number of arrivals per simulation is the area under the curve. About $\frac{1}{2}(13 \times 10) \approx 65$ (or you could start dissecting the areas of rectangles, triangles, and a trapezoid...).

There are 5000 arrivals shown in the right hand plot, so about $\left(\frac{5000}{65}\right) \approx 100$ simulations.

Correct Non-Stationary Arrival Techniques — Thinning Method

$a_0 \leftarrow 0$
$i \leftarrow 0$
repeat n `times (`
  $s \leftarrow a_i$
  repeat `(`
    $s \leftarrow s + Exponential\left(\frac{1}{\lambda_{\mathtt{max}}}\right)$
    $v \leftarrow Uniform(0, \lambda_{\mathtt{max}})$
  `)` **until** `(` $v < \lambda(s)$ `)`
  $a_{i+1} \leftarrow s$
  $i \leftarrow i + 1$
`)`
$a_1, a_2, \ldots, a_n$ `are arrival times`



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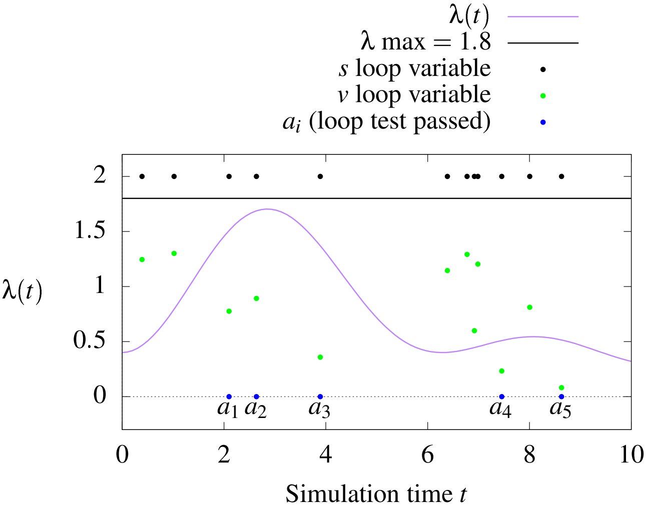

What is the optimal choice for $\lambda_{\mathtt{max}}$?

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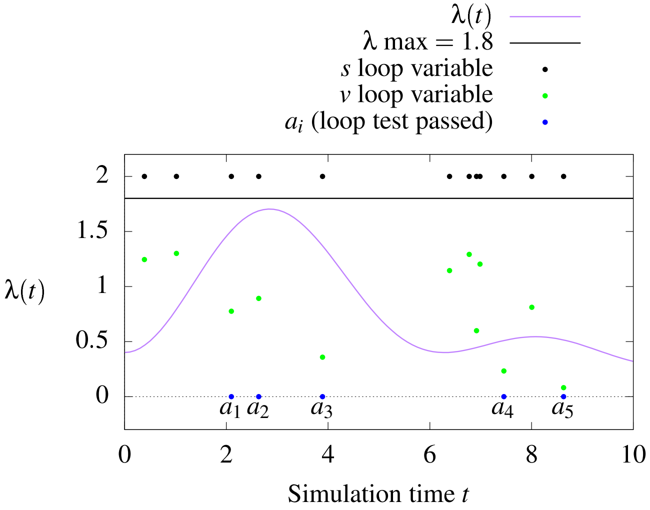

What is the optimal choice for $\lambda_{\mathtt{max}}$? $\max_t \lambda(t)$

How many times did the innermost loop instructions run for a_5?

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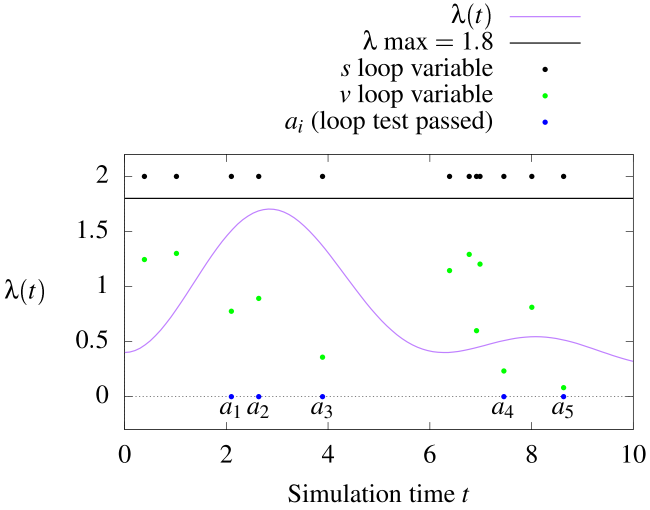


What is the optimal choice for $\lambda_{\mathtt{max}}$? $\max_t \lambda(t)$

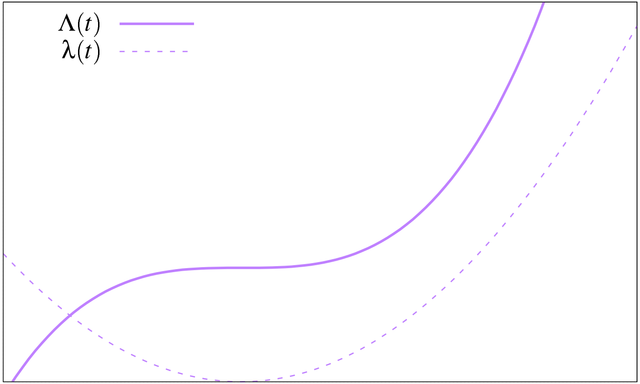
How many times did the innermost loop instructions run for a_5? 2

Let’s talk: lga-crv-applications …

Correct Non-Stationary Arrival Techniques — Inversion Method

Introducing the **Cumulative Arrival Function** $\Lambda(t)$

$$\Lambda(t) = \int_0^t \lambda(s)\ \mathrm{d}s$$



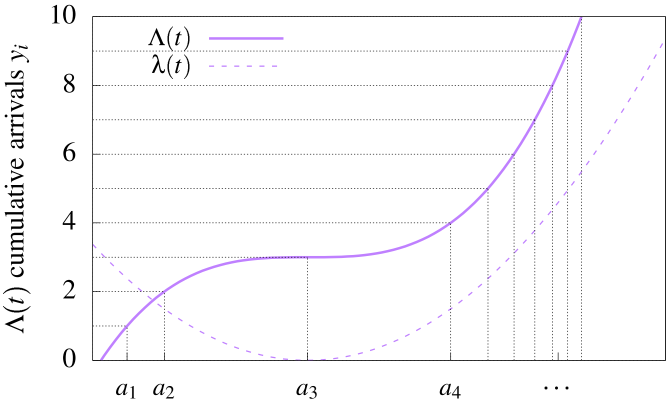
Correct Non-Stationary Arrival Techniques — Inversion Method

Motivating Geometry

If we take $n = 10$ evenly spaced points on the y-axis of $\Lambda(t)$, and solve

$$a_i = \Lambda^{-1}(y_i)$$

we see the distribution of a_i will reflect the arrival rates required by $\lambda(t)$.



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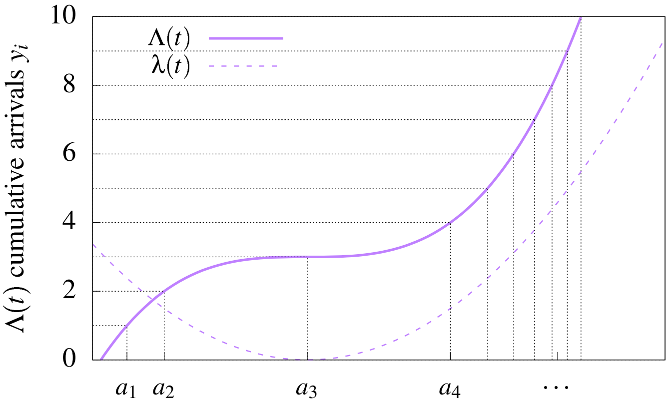

But this won't work for random arrival processes,
because there is **nothing random** in the technique (so far...).

Correct Non-Stationary Arrival Techniques — Inversion Method

Algorithm for n Randomized Arrivals

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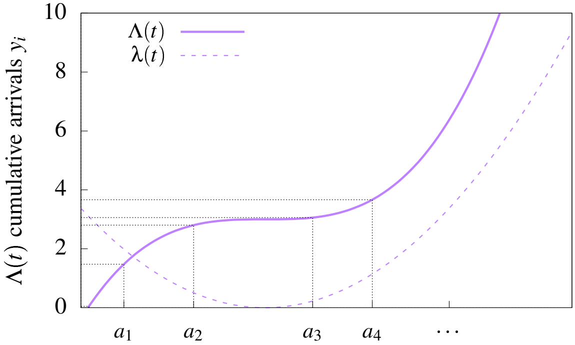

Non-Stationary Arrival Processes in NES

Unlike **the thinning method**, the $\Lambda^{-1}(t)$ method is not a form of **accept-reject**. So let's put it back into a form more suitable for the computational model...

Convert the inversion algorithm for non-stationary arrival times to a *single equation* for the **next arrival time** when you are given a_i the simulation time of the **current** event.

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"Next Arrival" equation for NES

$$a_{i+1} \leftarrow \Lambda^{-1}(\Lambda(a_i) + Exponential(1.0))$$

Special Case Derivation

Derive $\Lambda(t)$, $\Lambda^{-1}(y)$, and simplify the NES equation

$$a_{i+1} = \Lambda^{-1}\left(\Lambda(a_i) + Exponential(1.0)\right)$$

when $\lambda(t) = \alpha$ (a constant).

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when $\lambda(t) = \alpha$ (a constant).

1. Find $y = \Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)\ \mathrm{d}s = \int_0^t \alpha\ \mathrm{d}s = \alpha t$$

2. Find $t = \Lambda^{-1}(y)$:

$$y = \alpha t \quad \Rightarrow \quad t = \frac{y}{\alpha}$$

Special Case Derivation

2 Find $t = \Lambda^{-1}(y)$:

$$y = \alpha t \quad \Rightarrow \quad t = \frac{y}{\alpha}$$

3 Plug in and simplify

$$\begin{aligned}
a_{i+1} &= \Lambda^{-1}(\Lambda(a_i) + Exponential(1.0)) \\
\text{(use defn)} \quad &= \frac{1}{\alpha}(\Lambda(a_i) + Exponential(1.0)) \\
\text{(use defns)} \quad &= \frac{1}{\alpha}\left(\alpha a_i + -\frac{1}{1.0}\log(1-u)\right) \\
\text{(distribute)} \quad &= a_i + -\frac{1}{\alpha}\log(1-u) \\
\text{(use defn)} \quad &= a_i + Exponential(\frac{1}{\alpha})
\end{aligned}$$

Two Common Non-Stationary Patterns using Piecewise $\lambda(t)$s

The Coffee Shop

$$\lambda(t) = \begin{cases} \lambda_{06}(t) & 0600 \le t < 0900 \\ \lambda_{09}(t) & 0900 \le t < 1100 \\ \lambda_{11}(t) & 1100 \le t < 1300 \\ \lambda_{13}(t) & 1300 \le t < 1900 \end{cases}$$

What to do with arrival rate transitions with "$\lambda(t) = 0$ gaps" (eg 1900 $\to$ 0600)?

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i. Restart the arrival process algorithm at 6AM, or

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ii. Add a

$$\lambda_{19}(t) = 0 \;\; \text{for} \;\; 1900 \leq t < 0600$$

and treat as a "full cycle" arrival process…

24hr Emergency Room

$$\lambda(t) = \begin{cases} \lambda_{00}(t) & 0000 \leq t < 0300 \\ \lambda_{03}(t) & 0300 \leq t < 0500 \\ \lambda_{05}(t) & 0500 \leq t < 0900 \\ \lambda_{09}(t) & 0900 \leq t < 1100 \\ \lambda_{11}(t) & 1100 \leq t < 1500 \\ \lambda_{15}(t) & 1500 \leq t < 1900 \\ \lambda_{19}(t) & 1900 \leq t < 2200 \\ \lambda_{23}(t) & 2300 \leq t < 0000 \end{cases}$$

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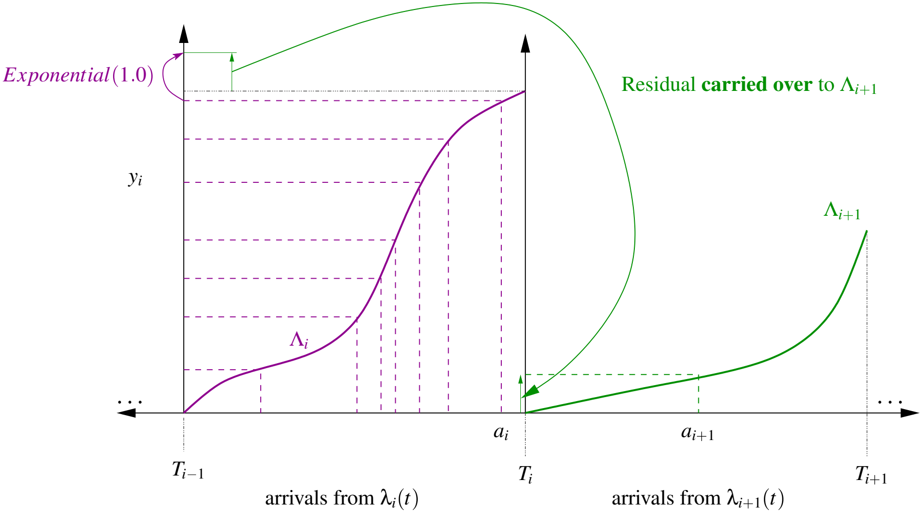
What to do with $\lambda_{09} \to \lambda_{11}$ transitions **without** intervening "gaps"?

What to do with piecewise arrival rates without "gaps"?

Inversion method: Carry the residual portion of $Exponential(1.0)$ over from Λ_i to Λ_{i+1}



$$a_{i+1} = \Lambda_{i+1}^{-1}\left(\Lambda_i(a_i) + Exponential(1.0) - \Lambda_i(T_i)\right)$$

What to do with piecewise arrival rates without “gaps”?

Thinning method: use a ”global” λ_{max}:

$$\lambda_{max} = \max_{\lambda_i} \{ \max_{t} \lambda_i(t) \}$$

(Less wasteful thinning methods exist, but are more complicated to explain.)

Also, **beware of thinning** when $\lambda_{\mathtt{max}}$ is large and there are regions of $\lambda(t) = 0 \ldots$