Stationary Arrival Processes

- Generated using $Exponential(\mu)$ interarrival times
- $\mu = \frac{1}{\lambda}$ where $\lambda = \frac{n}{t}$, average arrivals per time unit
- Also called a homogeneous arrival process
- The **next arrival equation** is

$$a_{i+1} = a_i + Exponential\left(\frac{1}{\lambda}\right)$$

Generate n Arrival Times

we don't really do this in a simulation!

$a_0 \leftarrow 0$
$i \leftarrow 0$
repeat `n times (`
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return $a_1, a_2, \ldots, a_n$

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A convenient fiction, arrival processes are often **time** dependent: $\lambda(t)$.

Non-Stationary Arrival Processes

Let the average arrival rate vary with t, you are given $\lambda(t)$.

Convert this stationary arrival equation to a non-stationary one:

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(where t is the **simulation clock**)

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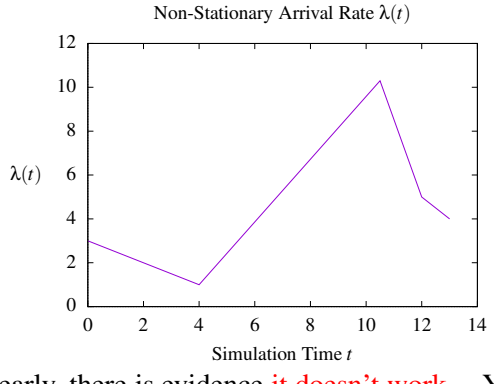
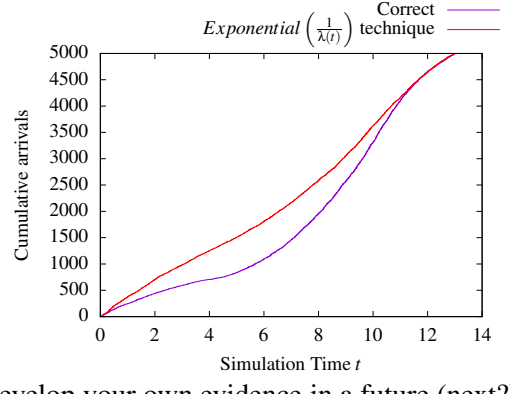
(where t is the **simulation clock**)

But does this technique actually work?

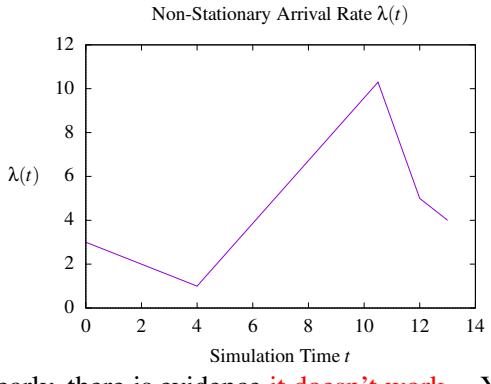
Compare Cumulative Arrivals between two techniques

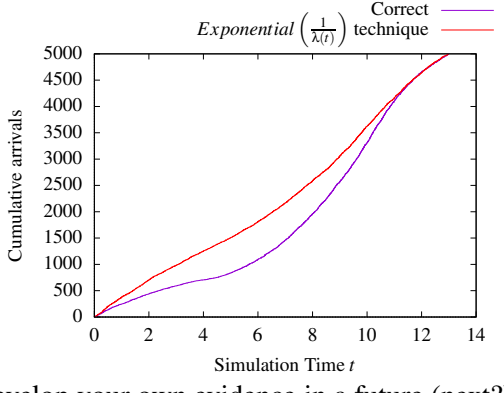




Clearly, there is evidence it doesn't work. . . You'll develop your own evidence in a future (next?) LGA.

Compare Cumulative Arrivals between two techniques

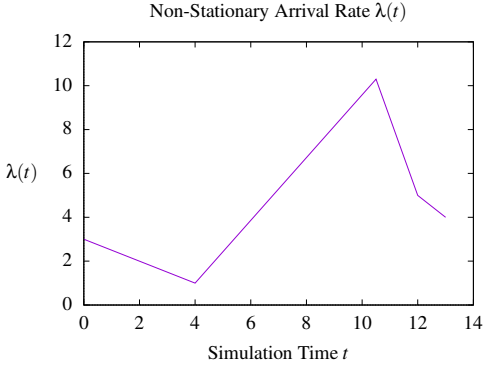


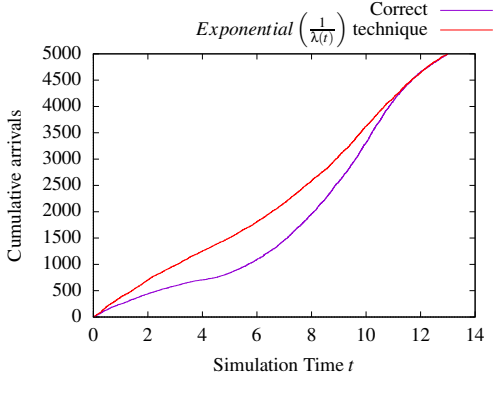


Clearly, there is evidence it doesn't work... You'll develop your own evidence in a future (next?) LGA.

Take 60s and discuss in your group — about how many simulations were required to generate the cumulative arrival plot on the right? All the information is on this screen...

Compare Cumulative Arrivals between two techniques



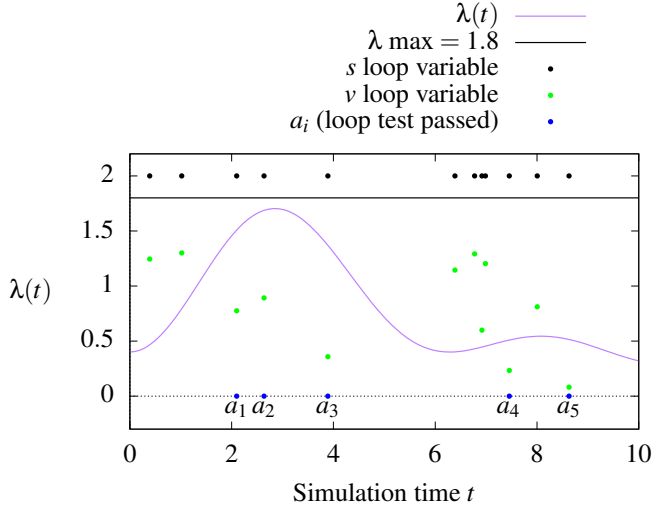


The left hand plot shows a **rate vs time relationship** — so the expected number of arrivals per simulation is the area under the curve. About $\frac{1}{2}(13 \times 10) \approx 65$ (or you could start dissecting the areas of rectangles, triangles, and a trapezoid...).

There are 5000 arrivals shown in the right hand plot, so about $\left(\frac{5000}{65}\right) \approx 100$ simulations.

Correct Non-Stationary Arrival Techniques — Thinning Method

$a_0 \leftarrow 0$
$i \leftarrow 0$
repeat n times (
  $s \leftarrow a_i$
  repeat (
    $s \leftarrow s + Exponential\left(\frac{1}{\lambda_{\max}}\right)$
    $v \leftarrow Uniform(0, \lambda_{\max})$
) **until** ($v < \lambda(s)$)
  $a_{i+1} \leftarrow s$
  $i \leftarrow i + 1$
)
$a_1, a_2, \ldots, a_n$ are arrival times



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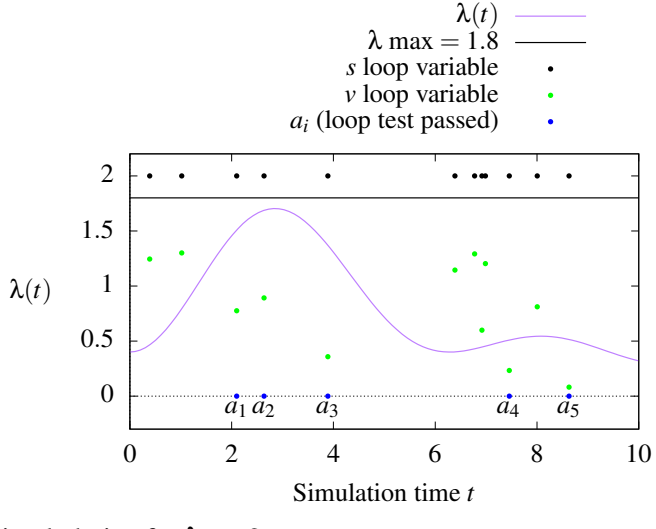

What is the optimal choice for $\lambda_{\max}$?

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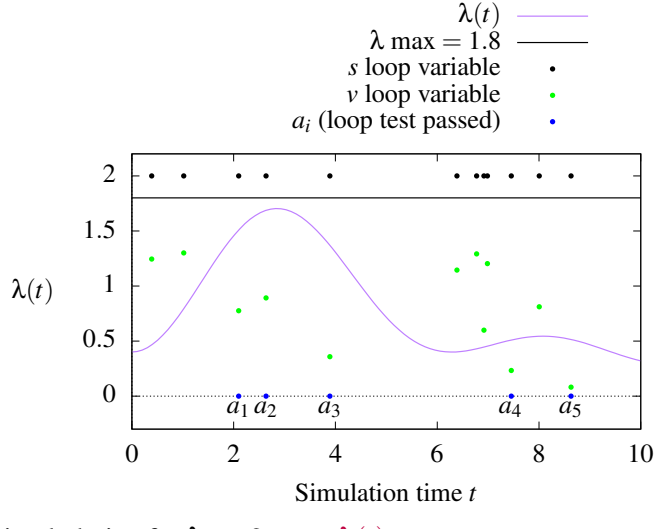

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How many times did the innermost loop instructions run for a_5?

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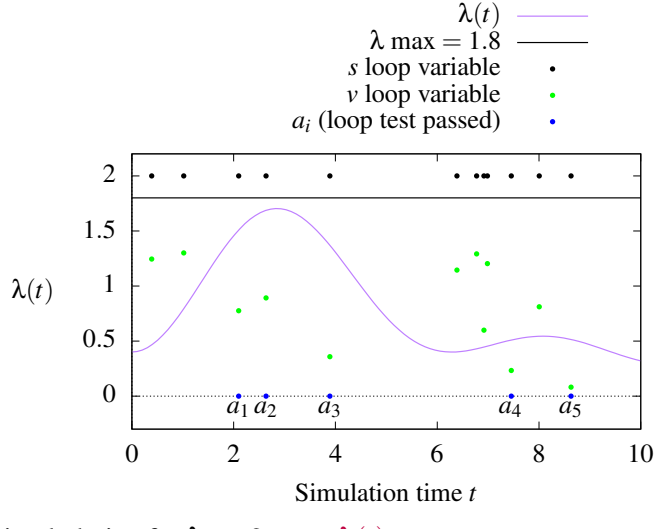


What is the optimal choice for $\lambda_{\max}$? $\max_t \lambda(t)$

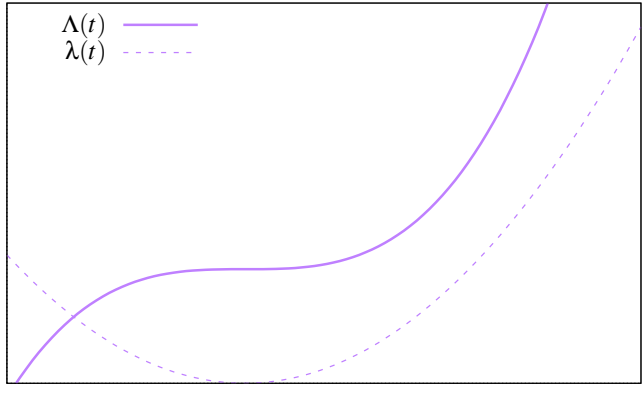
How many times did the innermost loop instructions run for a_5? 2

Let's talk: lga-crv-applications ...

Correct Non-Stationary Arrival Techniques — Inversion Method

Introducing the **Cumulative Arrival Function** $\Lambda(t)$

$$\Lambda(t) = \int_0^t \lambda(s)\ \mathrm{d}s$$



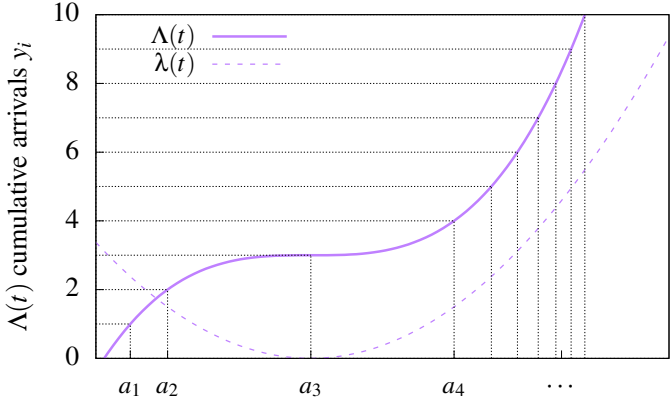
Correct Non-Stationary Arrival Techniques — Inversion Method

Motivating Geometry

If we take $n = 10$ evenly spaced points on the y-axis of $\Lambda(t)$, and solve

$$a_i = \Lambda^{-1}(y_i)$$

we see the distribution of a_i will reflect the arrival rates required by $\lambda(t)$.



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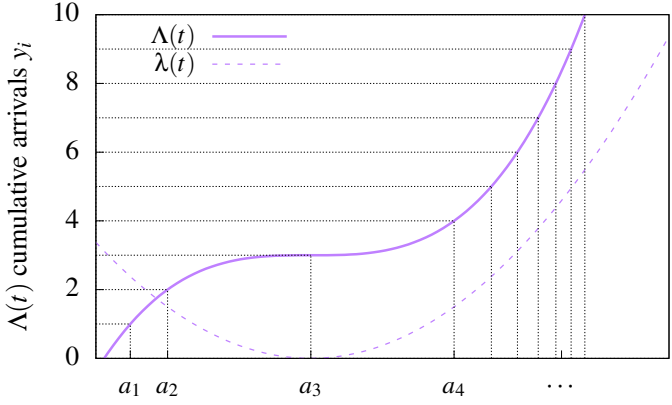

But this won't work for random arrival processes,
because there is **nothing random** in the technique (so far...).

Correct Non-Stationary Arrival Techniques — Inversion Method

Algorithm for n Randomized Arrivals

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repeat n `times (`
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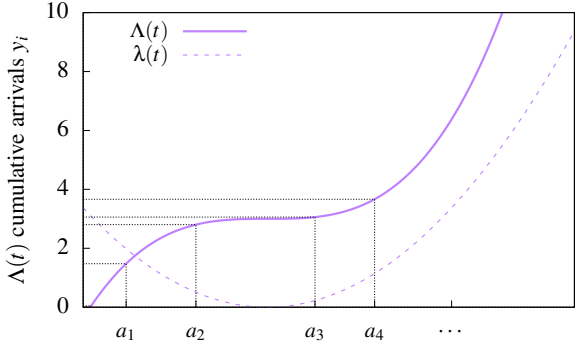

Non-Stationary Arrival Processes in NES

Unlike **the thinning method**, the $\Lambda^{-1}(t)$ method is not a form of **accept-reject**. So let's put it back into a form more suitable for the computational model...

Convert the inversion algorithm for non-stationary arrival times to a *single equation* for the **next arrival time** when you are given a_i the simulation time of the **current** event.

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"Next Arrival" equation for NES

$a_{i+1} \leftarrow \Lambda^{-1}(\Lambda(a_i) + Exponential(1.0))$

Special Case Derivation

Derive $\Lambda(t)$, $\Lambda^{-1}(y)$, and simplify the NES equation

$$a_{i+1} = \Lambda^{-1}\left(\Lambda(a_i) + Exponential(1.0)\right)$$

when $\lambda(t) = \alpha$ (a constant).

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when $\lambda(t) = \alpha$ (a constant).

1. Find $y = \Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)\ \mathrm{d}s = \int_0^t \alpha\ \mathrm{d}s = \alpha t$$

2. Find $t = \Lambda^{-1}(y)$:

$$y = \alpha t \quad \Rightarrow \quad t = \frac{y}{\alpha}$$

Special Case Derivation

2 Find $t = \Lambda^{-1}(y)$:

$$y = \alpha t \quad \Rightarrow \quad t = \frac{y}{\alpha}$$

3 Plug in and simplify

$$\begin{aligned}
a_{i+1} &= \Lambda^{-1}\left(\Lambda(a_i) + Exponential(1.0)\right) \\
\text{(use defn)} &= \frac{1}{\alpha}\left(\Lambda(a_i) + Exponential(1.0)\right) \\
\text{(use defns)} &= \frac{1}{\alpha}\left(\alpha a_i + -\frac{1}{1.0}\log(1-u)\right) \\
\text{(distribute)} &= a_i + -\frac{1}{\alpha}\log(1-u) \\
\text{(use defn)} &= a_i + Exponential(\frac{1}{\alpha})
\end{aligned}$$

Two Common Non-Stationary Patterns using Piecewise $\lambda(t)$s

The Coffee Shop

$$\lambda(t) = \begin{cases} \lambda_{06}(t) & 0600 \leq t < 0900 \\ \lambda_{09}(t) & 0900 \leq t < 1100 \\ \lambda_{11}(t) & 1100 \leq t < 1300 \\ \lambda_{13}(t) & 1300 \leq t < 1900 \end{cases}$$

What to do with arrival rate transitions with
"$\lambda(t) = 0$ gaps" (eg $1900 \rightarrow 0600$)?

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i. Restart the arrival process algorithm at 6AM, or

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What to do with arrival rate transitions with "$\lambda(t) = 0$ gaps" (eg $1900 \to 0600$)?

i. Restart the arrival process algorithm at 6AM, or

ii. Add a

$$\lambda_{19}(t) = 0 \text{ for } 1900 \le t < 0600$$

and treat as a "full cycle" arrival process...

24hr Emergency Room

$$\lambda(t) = \begin{cases} \lambda_{00}(t) & 0000 \le t < 0300 \\ \lambda_{03}(t) & 0300 \le t < 0500 \\ \lambda_{05}(t) & 0500 \le t < 0900 \\ \lambda_{09}(t) & 0900 \le t < 1100 \\ \lambda_{11}(t) & 1100 \le t < 1500 \\ \lambda_{15}(t) & 1500 \le t < 1900 \\ \lambda_{19}(t) & 1900 \le t < 2200 \\ \lambda_{23}(t) & 2300 \le t < 0000 \end{cases}$$

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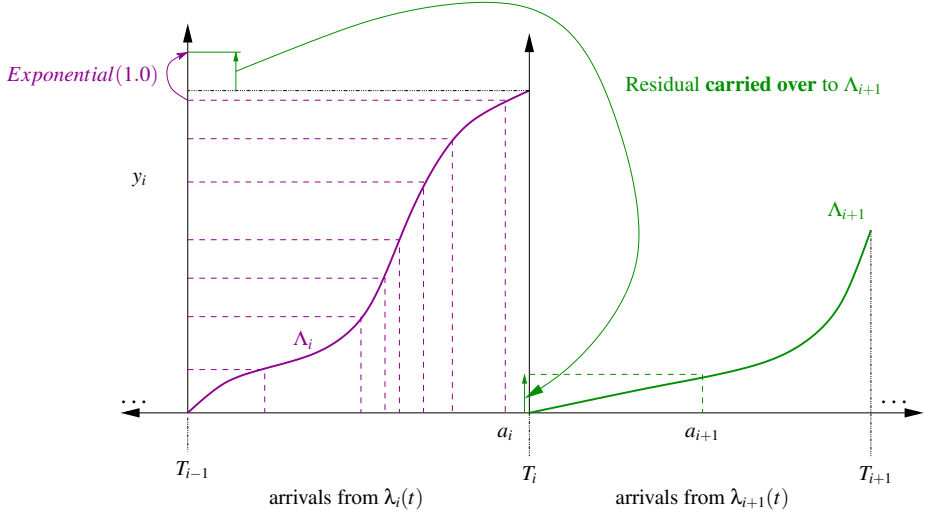
What to do with $\lambda_{09} \to \lambda_{11}$ transitions **without** intervening "gaps"?

What to do with piecewise arrival rates without "gaps"?

Inversion method: Carry the residual portion of $Exponential(1.0)$ over from Λ_i to Λ_{i+1}



$$a_{i+1} = \Lambda_{i+1}^{-1}\left(\Lambda_i(a_i) + Exponential(1.0) - \Lambda_i(T_i)\right)$$

What to do with piecewise arrival rates without "gaps"?

Thinning method: use a "global" $\lambda_{\max}$:

$$\lambda_{\max} = \max_{\lambda_i}\{\max_{t}\lambda_i(t)\}$$

(Less wasteful thinning methods exist, but are more complicated to explain.)

Also, **beware of thinning** when $\lambda_{\texttt{max}}$ is large and there are regions of $\lambda(t) = 0\ldots$