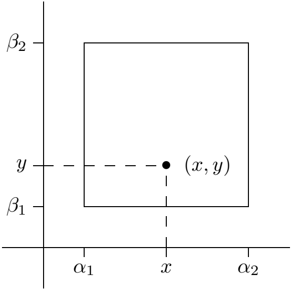
Random Points

September 25, 2025

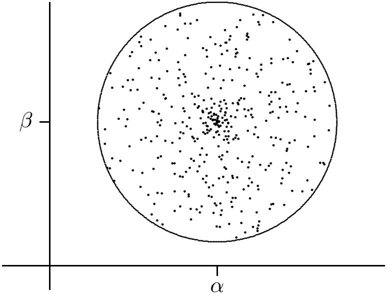
- Generate a point at random inside a rectangle with opposite corners at (α_1, β_1) and (α_2, β_2)




```
x = Uniform(α1, α2);
y = Uniform(β1, β2);
```

Example 2.3.8

- Generate a point (x, y) at random *interior* to the circle of radius ρ centered at (α, β)



θ = Uniform($-\pi$, π);
r = Uniform(0, ρ);
x = α + r * cos(θ);
y = β + r * sin(θ);

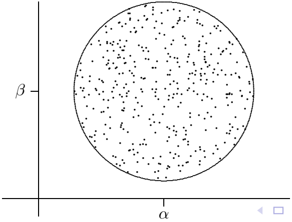
INCORRECT!

Acceptance/Rejection

- Generate a point at random within a circumscribed square and then either accept or reject the point

Generating a Random Point

```
do {
    x = Uniform(-ρ, ρ);
    y = Uniform(-ρ, ρ);
} while (x * x + y * y >= ρ * ρ);
x = α + x;
y = β + y;
return (x, y);
```



Alternatives to Accept/Reject (Deriving a Radial Distribution...)

Not always possible, and not always the most straight-forward, sometimes a little math gets you out of the Accept/Reject pit.

$$\text{Circumference}\big|_{\text{x}} = 2\pi\text{x} \qquad \text{Circumference}\big|_{2\text{x}} = 4\pi\text{x}$$

We want $f(x)$, the distribution of points along the circumference of radius $2x$ to be twice as much as at x. For some $C > 0$ we expect $f(x) = Cx$, what is C?

$$1 \equiv \int_0^R f(x)\ \mathrm{d}x = \int_0^R Cx\ \mathrm{d}x = C\left[\frac{1}{2}x^2\right]_0^R = C\frac{R^2}{2}$$

solving

$$\frac{CR^2}{2} = 1 \quad \Rightarrow \quad C = \frac{2}{R^2} \quad \Rightarrow \quad f(x) = \frac{2x}{R^2}$$

We have a pdf $f(x)$ for the distribution of points on the circumscribed annular rings... now what?

Deriving a Radial Distribution (continued)

Step 1: Integrate $f(x)$ to get the **cumulative distribution function**:

$$F(x) = \int_0^x f(t)\ \mathrm{d}t = \int_0^x \frac{2t}{R^2}\ \mathrm{d}t = \frac{2}{R^2}\left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{R^2}$$

Step 2: Clearly $0 < F(x) < 1$, let $u \leftarrow \mathit{Random}()$ and set them equal to each other, solve for x which is back in the **domain** of $F(x)$ and $f(x)\dots$

$$u = \frac{x^2}{R^2} \quad \Rightarrow \quad x^2 = uR^2 \quad \Rightarrow \quad x = R\sqrt{u} = F^{-1}(u)$$

From the geometry $0 < x < R$ so we don't need $|\cdot|$ sign pedantics.

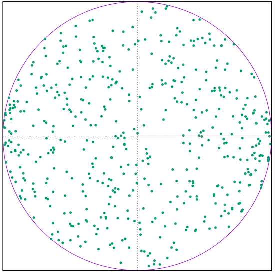
Now we can randomize points in a circle with two draws from `Random()`:

$$\theta = \mathrm{Uniform}(0, 2\pi) \qquad \mathrm{r} = \mathrm{R}\sqrt{(\mathrm{Random}())}$$

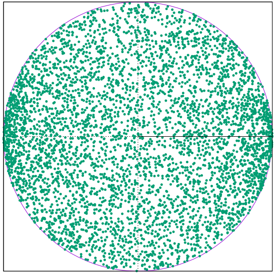
$$x = r\cos\theta \qquad y = r\sin\theta$$

Stochastic or Suspect?

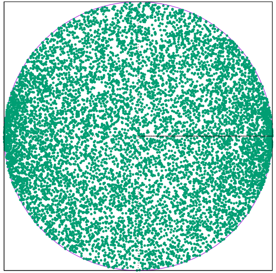
$n = 500$ random points in a circle



$n = 5000$ random points in a circle



$n = 10000$ random points in a circle



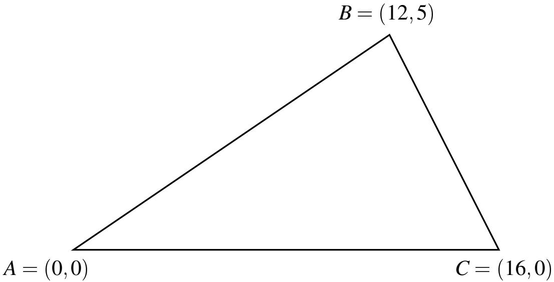
What do you think of these “random points” in a circle?
Can you speculate what the (flawed) algorithm is that generated these points?

Alternatives to Accept/Reject

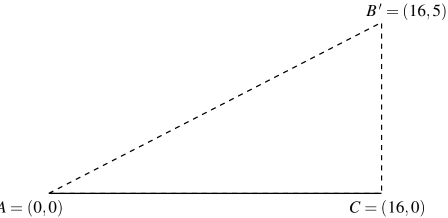
Suppose you had a triangle at coordinates

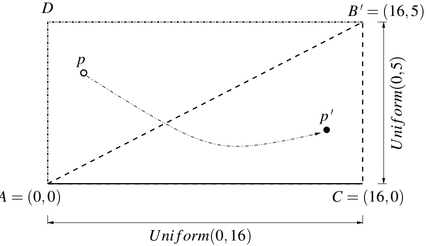
$$A(0,0) \qquad B(12,5) \qquad C(16,0)$$

how would you **uniformly** randomize a point inside the triangle with at most two draws from your pRNG?



Strategy A: Shearing to a known solution





Strategy A:

- Shear point B from $(12,5)$ to $B'(16,5)$
- Choose random point in the A, $(0,5)$, B', C rectangle
- If the point is above $\overline{AB'}$, reflect it (carefully!) to the other side
- Unshear

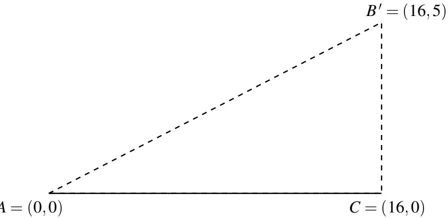
A shearing matrix that keeps y coords the same:

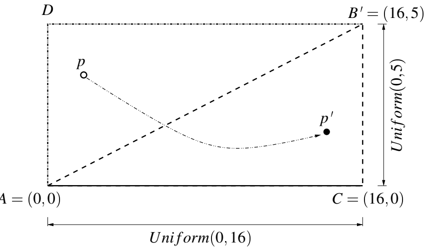
$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+sy \\ y \end{bmatrix}$$

Calculate s:

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix} \Rightarrow \quad 12+5s = 16 \Rightarrow \quad s = -\frac{4}{5}$$

Strategy A: Shearing to a known solution





Strategy A:

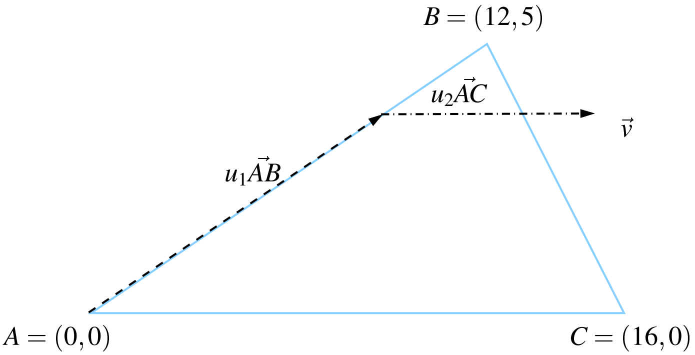
- **We must be careful with reflecting!**
- If we simply "flip" the points vertically over the hypotnuse $\overline{AB'}$, we will end up with just as many points near the little tip of the triangle as near the altitude — that can't be right.
- We must flip the point to its congruent location in $\triangle AB'C$.
- Given a randomized point $p = (i, j)$ and $j > \frac{5}{16}i$ we want the translated location to $p'(i', j') = (16 - i, 5 - j)$.

Strategy B: Vector Addition

- Choose points with vector math (u_1 and u_2 from $Random()$)

$$\vec{v} = u_1\vec{AB} + u_2\vec{AC}$$

- If $\vec{v}$ lies beyond $\overline{BC}$, **carefully** reflect it back into $\triangle ABC$.



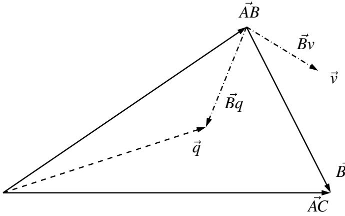
Strategy B: Vector Addition (cross product approach)

- Choose points with vector math (u_1 and u_2 from $Random()$)

$$\vec{v} = u_1\vec{AB} + u_2\vec{AC}$$

- If $\vec{v}$ lies beyond $\overline{BC}$, **carefully** reflect it back into $\triangle ABC$.

$$\vec{v}\,' = \vec{AB} + \vec{AC} - \vec{v}$$



How to tell if the point is beyond $\overline{BC}$? **Remember the right hand rule of cross products!** Let $\vec{q} = \frac{1}{4}(\vec{AB} + \vec{AC})$, then look to see if the **sign** of the cross products

$$\vec{BC} \times \vec{Bq} \quad \text{and} \quad \vec{BC} \times \vec{Bv}$$

match. If they do, then $\vec{v}$ lies in the triangle, otherwise reflect it back.

Recall, for $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$,

$$\vec{a} \times \vec{b} = (a_1 b_2 - a_2 b_1)\vec{k}$$

where $\vec{k}$ is the unit vector perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$.

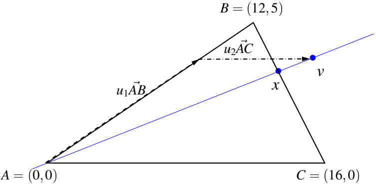
Strategy B: Vector Addition (line intersection approach)

- Choose points with vector math (u_1 and u_2 from $Random()$)

$$\vec{v} = u_1\vec{AB} + u_2\vec{AC}$$

- If $\vec{v}$ lies beyond $\overline{BC}$, **carefully** reflect it back into $\triangle ABC$.

$$\vec{v}\,' = \vec{AB} + \vec{AC} - \vec{v}$$



How to tell if the point is beyond $\overline{BC}$?
Find x the intersection of the line containing A and v with the line BC.

Compare the distance from A to v and from A to x.