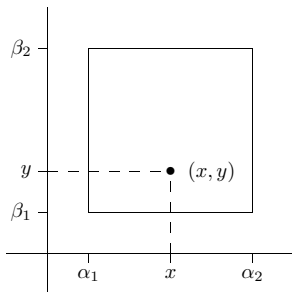


# Random Points

September 17, 2026

# Geometric Applications

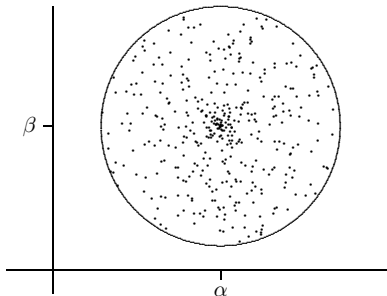
- Generate a point at random inside a rectangle with opposite corners at  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$



```
x = Uniform( $\alpha_1$ ,  $\alpha_2$ );  
y = Uniform( $\beta_1$ ,  $\beta_2$ );
```

## Example 2.3.8

- Generate a point  $(x, y)$  at random *interior* to the circle of radius  $\rho$  centered at  $(\alpha, \beta)$



```
 $\theta = \text{Uniform}(-\pi, \pi);$   
 $r = \text{Uniform}(0, \rho);$   
 $x = \alpha + r * \cos(\theta);$   
 $y = \beta + r * \sin(\theta);$ 
```

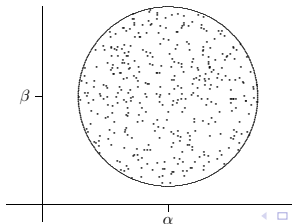
**INCORRECT!**

# Acceptance/Rejection

- Generate a point at random within a circumscribed square and then either accept or reject the point

## Generating a Random Point

```
do {  
     $x = \text{Uniform}(-\rho, \rho);$   
     $y = \text{Uniform}(-\rho, \rho);$   
} while ( $x * x + y * y \geq \rho * \rho$ );  
 $x = \alpha + x;$   
 $y = \beta + y;$   
return ( $x, y$ );
```



## Alternatives to Accept/Reject (Deriving a Radial Distribution...)

**Not always possible, and not always the most straight-forward, sometimes a little math gets you out of the Accept/Reject pit.**

$$\text{Circumference}|_x = 2\pi x \quad \text{Circumference}|_{2x} = 4\pi x$$

We want  $f(x)$ , the distribution of points along the circumference of radius  $2x$  to be twice as much as at  $x$ . For some  $C > 0$  we expect  $f(x) = Cx$ , what is  $C$ ?

$$1 \equiv \int_0^R f(x) \, dx = \int_0^R Cx \, dx = C \left[ \frac{1}{2}x^2 \right]_0^R = C \frac{R^2}{2}$$

solving

$$\frac{CR^2}{2} = 1 \quad \Rightarrow \quad C = \frac{2}{R^2} \quad \Rightarrow \quad f(x) = \frac{2x}{R^2}$$

We have a pdf  $f(x)$  for the distribution of points on the circumscribed annular rings... now what?

## Deriving a Radial Distribution (continued)

**Step 1:** Integrate  $f(x)$  to get the **cumulative distribution function**:

$$F(x) = \int_0^x f(t) \, dt = \int_0^x \frac{2t}{R^2} \, dt = \frac{2}{R^2} \left[ \frac{t^2}{2} \right]_0^x = \frac{x^2}{R^2}$$

**Step 2:** Clearly  $0 < F(x) < 1$ , let  $u \leftarrow \text{Random}()$  and set them equal to each other, solve for  $x$  which is back in the **domain** of  $F(x)$  and  $f(x)$ ...

$$u = \frac{x^2}{R^2} \quad \Rightarrow \quad x^2 = uR^2 \quad \Rightarrow \quad x = R\sqrt{u} = F^{-1}(u)$$

From the geometry  $0 < x < R$  so we don't need  $|\cdot|$  sign pedantics.

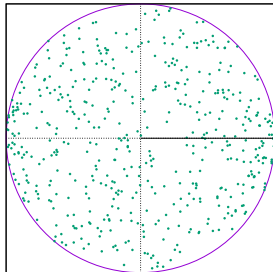
Now we can randomize points in a circle with two draws from  $\text{Random}()$ :

$$\theta = \text{Uniform}(0, 2\pi) \quad r = R\sqrt{\text{Random}()}$$

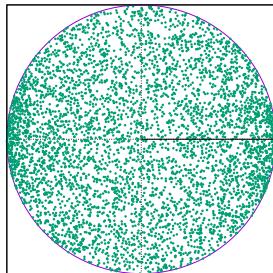
$$x = r \cos \theta \quad y = r \sin \theta$$

# Stochastic or Suspect?

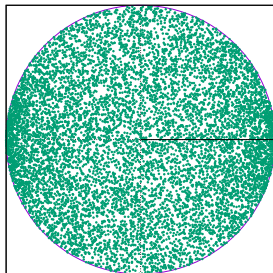
$n = 500$  random points in a circle



$n = 5000$  random points in a circle



$n = 10000$  random points in a circle



What do you think of these “random points” in a circle?

Can you speculate what the (flawed) algorithm is that generated these points?

## Alternatives to Accept/Reject

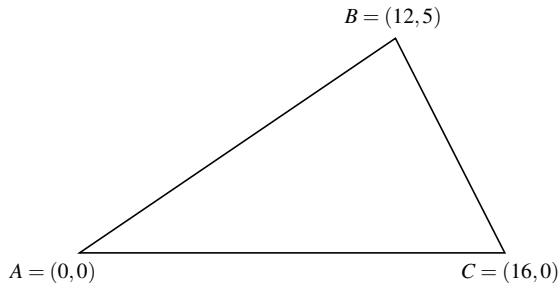
Suppose you had a triangle at coordinates

$A(0,0)$

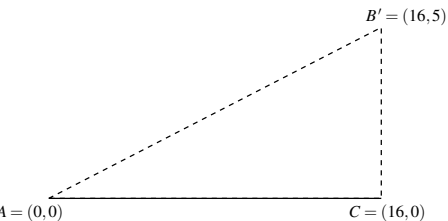
$B(12,5)$

$C(16,0)$

how would you **uniformly** randomize a point inside the triangle with at most two draws from your pRNG?



## Strategy A: Shearing to a known solution



### Strategy A:

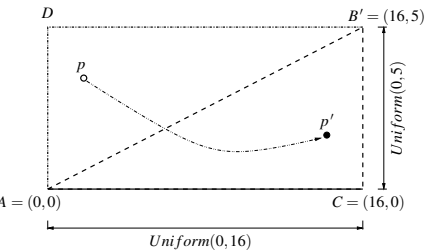
- Shear point  $B$  from  $(12,5)$  to  $B'(16,5)$
- Choose random point in the  $A, (0,5), B', C$  rectangle
- If the point is above  $\overline{AB'}$ , reflect it (carefully!) to the other side
- Unshear

A shearing matrix that keeps  $y$  coords the same:

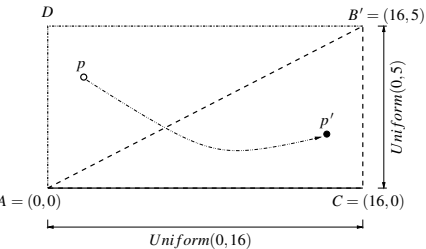
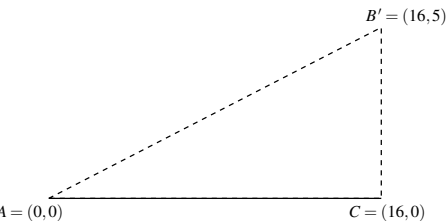
$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

Calculate  $s$ :

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix} \Rightarrow 12 + 5s = 16 \Rightarrow s = -\frac{4}{5}$$



## Strategy A: Shearing to a known solution



### Strategy A:

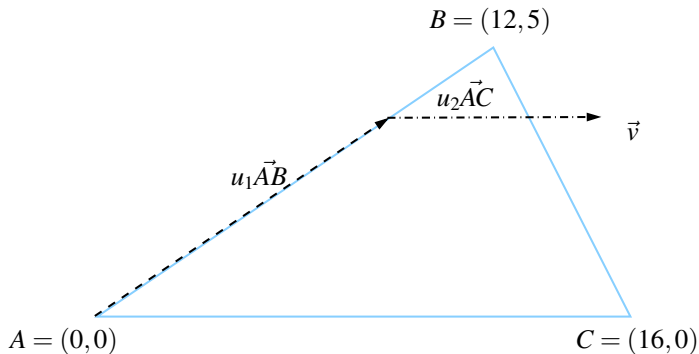
- ▶ **We must be careful with reflecting!**
- ▶ If we simply “flip” the points vertically over the hypotenuse  $\overline{AB'}$ , we will end up with just as many points near the little tip of the triangle as near the altitude — that can’t be right.
- ▶ We must flip the point to its congruent location in  $\triangle AB'C$ .
- ▶ Given a randomized point  $p = (i, j)$  and  $j > \frac{5}{16}i$  we want the translated location to  $p'(i', j') = (16 - i, 5 - j)$ .

## Strategy B: Vector Addition

- Choose points with vector math ( $u_1$  and  $u_2$  from  $Random()$ )

$$\vec{v} = u_1 \vec{AB} + u_2 \vec{AC}$$

- If  $\vec{v}$  lies beyond  $\overline{BC}$ , **carefully** reflect it back into  $\triangle ABC$ .



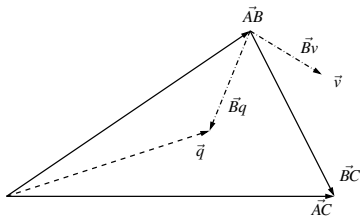
## Strategy B: Vector Addition (cross product approach)

- Choose points with vector math ( $u_1$  and  $u_2$  from  $Random()$ )

$$\vec{v} = u_1 \vec{AB} + u_2 \vec{AC}$$

- If  $\vec{v}$  lies beyond  $\overline{BC}$ , **carefully** reflect it back into  $\triangle ABC$ .

$$\vec{v}' = \vec{AB} + \vec{AC} - \vec{v}$$



How to tell if the point is beyond  $\overline{BC}$ ? **Remember the right hand rule of cross products!** Let  $\vec{q} = \frac{1}{4}(\vec{AB} + \vec{AC})$ , then look to see if the **sign** of the cross products

$$\vec{BC} \times \vec{Bq} \quad \text{and} \quad \vec{BC} \times \vec{Bv}$$

match. If they do, then  $\vec{v}$  lies in the triangle, otherwise reflect it back.

Recall, for  $\vec{a} = (a_1, a_2)$  and  $\vec{b} = (b_1, b_2)$ ,

$$\vec{a} \times \vec{b} = (a_1 b_2 - a_2 b_1) \vec{k}$$

where  $\vec{k}$  is the unit vector perpendicular to the plane spanned by  $\vec{a}$  and  $\vec{b}$ .

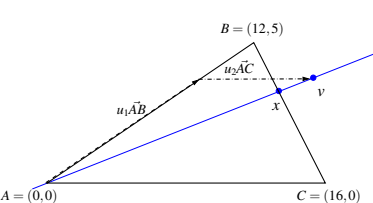
## Strategy B: Vector Addition (line intersection approach)

- Choose points with vector math ( $u_1$  and  $u_2$  from  $Random()$ )

$$\vec{v} = u_1 \vec{AB} + u_2 \vec{AC}$$

- If  $\vec{v}$  lies beyond  $\overline{BC}$ , **carefully** reflect it back into  $\triangle ABC$ .

$$\vec{v}' = \vec{AB} + \vec{AC} - \vec{v}$$



How to tell if the point is beyond  $\overline{BC}$ ?

Find  $x$  the intersection of the line containing  $A$  and  $v$  with the line  $BC$ .

Compare the distance from  $A$  to  $v$  and from  $A$  to  $x$ .