The Need for pRNG Streams

Recall the *Simple Inventory System* where demands for inventory items slowly depletes an initial maximum store of S. Periodically, inventory size l is evaluated and if $l < s$ a restocking order is placed for $S - l$ items.

Demands placed when $l < 0$ incur a penalty for the seller, so keeping $l > 0$ should be correlated with increased profits. But each order has an overhead cost diminishing profit — the optimal re-ordering policy (s) is not obvious in this case.

Additionally, we should consider non-instaneous delivery of ordered items, aka *delivery lag*.

We want to assess the difference in two different re-ordering policies s_A and s_B

A Naive Approch to s_A vs s_B

The `SIM` will use a sequence of random values in two different **stochastic processes** in the simulation: **demands** and **delivery lag**.

```
$ cat SIM-AvsB.sh
set -e
for seed in $(cat SEEDS) ; do
    # collect policy specific results for a given random seed
    ./SIM A ${seed} >A-output-${seed}.out
    ./SIM B ${seed} >B-output-${seed}.out
done
$ ./ANALYZE.sh A-output-*.out -- B-output-*.out
n=500 mean=0.266 popstddev=1.530 ci90=(0.230,0.302)
```

Clearly, we detected a difference — but is this accurate? Does it really exist? Is the claimed confidence interval correct?

Discuss. . .

A Naive Approch to s_A vs s_B

That was a trick question — the answer to both is **YES**. It's a statistical test of independent samples, the only "claim" the confidence interval asserts is that

> *If we ran the experiment,* ***with a different set of seeds*** *we expect the mean difference between the two scenarios to fall within* $(0.230, 0.320)$ *about 90% of the time.*

The **real** question is **qualatitive**:

> WAS THE *experimental design* OPTIMIZED TO MEASURE THE **MOST ACCURATE** DIFFERENCE IN POLICIES s_A AND s_B?

A Naive Approch to s_A vs s_B

No.

The re-order polices will move around the (simultation time) of reorders. The demand seen by s_A and s_B will be the same **up until one of the policies places an order**. The policy that orders first (suppose it is s_A) will use a draw from the pRNG (call it x_k) to determine delivery lag.

But x_k will produce a demand for s_B. And some time after this, s_B will use an x_m, $m > k$ to determine a delivery lag — but most likely x_m will generate a demand arrival for s_A.

A Naive Approch to s_A vs s_B

As the two policies **diverge** from their initial syncronized use of the pRNG sequence $x_0, x_1, \ldots$, they will see different patterns in both **demands** and **delivery lags**.

Our naive comparison of the two policies **is not flawed** — but we can perform a more sensitive experiment using **independent sequences of random values** for our two stochastic processes in the simulation: **demands**, and **delivery lags**.
In this way, the same demand profile is seen by both s_A and s_B experiments, and both experiments see the same sequence of random delivery lags.

This is an example of ***Variance Reduction*** in experimental design.

Problematic Solutions

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   Not much better, the quality of pRNGs is an assessment of their apparent randomness *internally* within their own generated values (for a particular seed). There have been **no studies** showing that two different pRNGs are random and indpendent of each other.

   Why not? Because this is universally considered the wrong use of pRNGs in simulations.

Do the Right Thing: pRNG (sub) Streams

We can mathematically break of up the **whole sequence** of pRNG generated values into disjoint subsequences.

These are called ***streams*** of the pRNG.

How? Suppose you want to break up an (a, m) Lehmer generator into 5 distinct streams (indexed by j):

$$\text{elements per streams} = e = \left\lfloor \frac{m}{5} \right\rfloor$$

Do the Right Thing: pRNG (sub) Streams

Given a seed x_0, the $j = 0$ stream begins at $x_0 = x_{0,0}$. The $j = 1$ stream begins at

$$x_{1,0} = a^e x_0 \bmod m$$

the $j = 2$ stream begins at

$$x_{2,0} = a^{2e} x_0 \bmod m$$

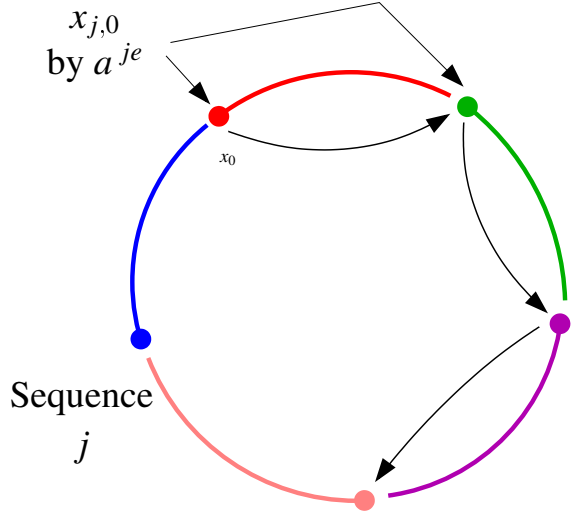
and the jth stream begins at

$$x_{j,0} = a^{je} x_0 \bmod m$$

The "next value" function remains the same:

$$f(x_{j,i}) = a x_{j,i-1} \bmod m$$

You just have j different states to store in memory.



A Better Approch to s_A vs s_B

```
$ cat SIM-AvsB-streams.sh
set -e
for seed in $(cat SEEDS) ; do
    # collect policy specific results for a given random seed
    # Use 2 streams, stream 0 for demands (-d), 1 for lags (-l)
    ./SIM A ${seed} --streams 2 -d 0 -l 1 >A-output-${seed}.out
    ./SIM B ${seed} --streams 2 -d 0 -l 1 >B-output-${seed}.out
done
$ ./ANALYZE.sh A-output-*.out -- B-output-*.out
n=500 mean=0.315 popstddev=0.352 ci90=(0.307,0.324)
```

Note the substantially better confidence interval due to ***variance reduction***

— improved by a factor of 18.

A Paired Approch to s_A vs s_B (Ideal)

```
$ cat SIM-AvsB-streams-paired.sh
set -e
for seed in $(cat SEEDS) ; do
    # collect policy specific results for a given random seed
    # Use 2 streams, stream 0 for demands (-d), 1 for lags (-l)
    ./SIM A ${seed} --streams 2 -d 0 -l 1 >A-output-${seed}.out
    ./SIM B ${seed} --streams 2 -d 0 -l 1 >B-output-${seed}.out
done
# a little more effort is required in ANALYZE-PAIRED.sh to pair
# same seed results together correctly
$ ./ANALYZE-PAIRED.sh "%s-output-%d.out" A B SEEDS
n=500 mean=0.353 popstddev=0.341 ci90=(0.328,0.378)
```

We treat the s_A vs s_B as a **placebo/treatment** or **pre/post** experimental design — the difference between same seed results are treated as the original data set δ_i.

By treating differences as accurately as possible in our statistical tests, we can again improve the result.

A Paired Approch to s_A vs s_B (Ideal)

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```

Using a paired experimental design **isn't usually considered an example of *variance reduction*** — it's simply best practice in statistical analysis.

You aren't **always guaranteed** of a better result vs. a non-paired analysis — but you do have the most accurate result!