# Formality of Expressions

Mathematical expressions are made up of two things: values and operators.

Values may be either variables (`x`, `stockPrice`) or literal values (3, 14, 3.14E2).

(Binary) operators are: addition, subtraction, multiplication (`*`), division (/), modulus (`%`), and exponentiation (`**`).

- Binary operators always take two arguments, conventionally called the **lhs** and **rhs**.

```
lhs + rhs
```

- All binary operators have two important properties: **precedence** and **associativity**.

# Mathematical Precedence

Faced with a programming language expression such as:

```
a * b + 3 ** x / 2
```

in what order are the operations performed?

# Mathematical Precedence

Faced with a programming language expression such as:

```
a * b + 3 ** x / 2
```

in what order are the operations performed?

| When? | Precedence | Operations |
|---|---|---|
| First | Highest | `**` (exponentiation) |
| | | `*  /  %` (multiplicative class) |
| Last | Lowest | `+  -` (additive class) |

We are sure you remember this as “**please excuse my dear aunt sally**.”

# Operator Associativity

Faced with a programming language expression such as:

```
a - b - c
```

in what order are the **same-precedence** operations performed? Yes, you already know this because it has been drilled into your brain along with "Aunt Sally": left-to-right evaluation!

```
(a - b) - c
```

Left-to-right evaluation comes from the **left-associative property** of addition and multiplicative class operations.

Why? Because the middle term `b` is *associated* with the operator *to its left*.

# Right Associativity

Left-associative operators are pretty common in both mathematics and programming, what are some **right-associative** operators?

**Exponentiation**

$$5^{x^y} \xrightarrow{\text{in programming}} \texttt{5 ** x ** y} \xrightarrow{\text{right associative}} \texttt{5 ** (x ** y)}$$

The **middle term** is associated with the operator **on its right**.

# Right Associativity

Left-associative operators are pretty common in both mathematics and programming, what are some **right-associative** operators?

**Assignment (=) in C, C++, Python, Java, . . .**

The following two code samples are equivilent, due to the right-associativity of value assignment

```
a = c = 3
```

```
c = 3
a = c
```

The **middle term** is associated with the operator **on its right**.

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`

We begin with an empty `valueStack` and `opStack`, and traverse the elements of an expression from left to right.

→ Top of Stacks → `Expression`

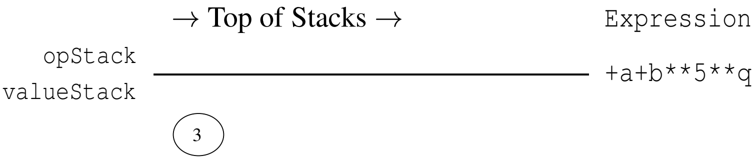
`opStack`

`valueStack` `3+a+b**5**q`

Begin. . .

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

`Expression`

`opStack`

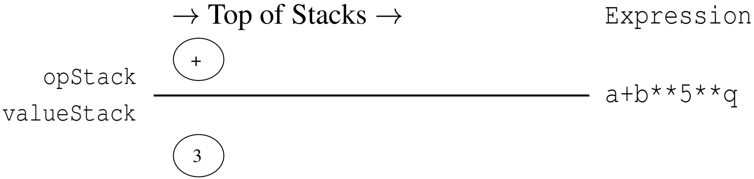
`+a+b**5**q`

`valueStack`

Push value `3` onto `valStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

Expression

opStack

+

valueStack

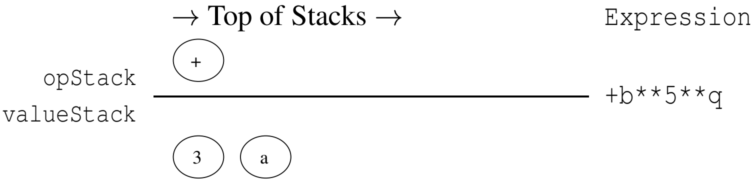
3

`a+b**5**q`

Push operation + onto `opStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

Expression

opStack

+

+b**5**q

valueStack

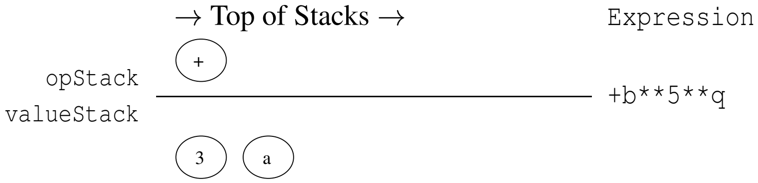
3

a

Push value a onto valStack

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



Expression's **left-associative** + precedence is $\leq$ + (opStack) precedence → pop the opStack

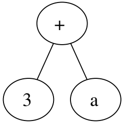
# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`

→ Top of Stacks → `Expression`

`opStack`

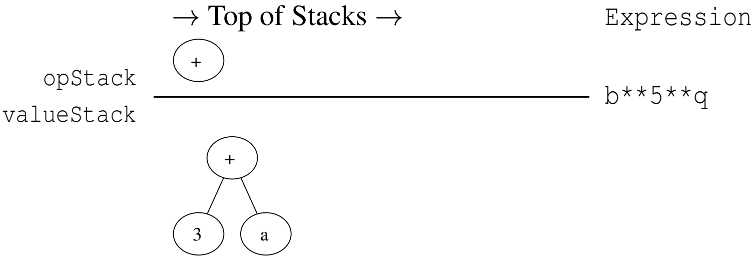
`valueStack` `+b**5**q`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

Expression

opStack

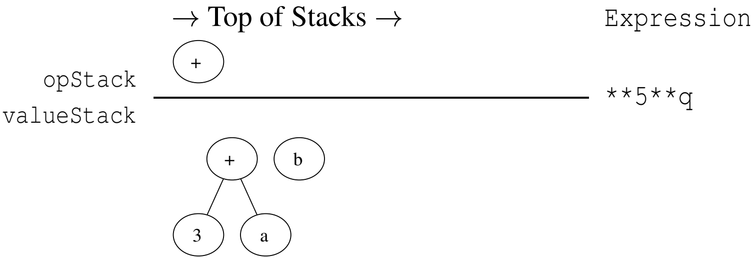
valueStack

`b**5**q`

Push operation + onto `opStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

`Expression`

`opStack`

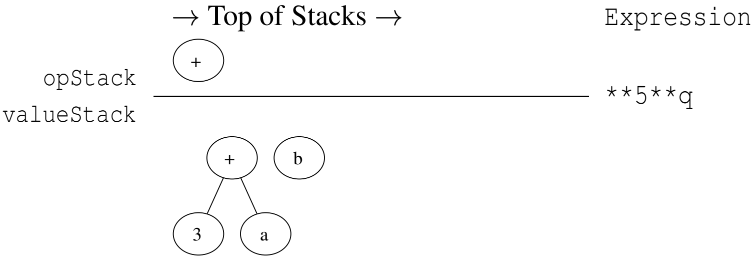
`valueStack`

`**5**q`

Push value `b` onto `valStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



Expression's **right-associative** `**` precedence is ≥ + (`opStack`) precedence

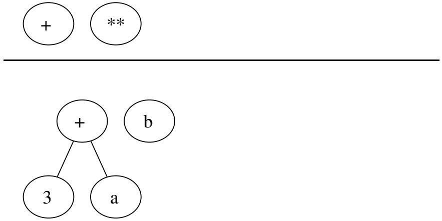
# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`

→ Top of Stacks → `Expression`

`opStack`

`valueStack`

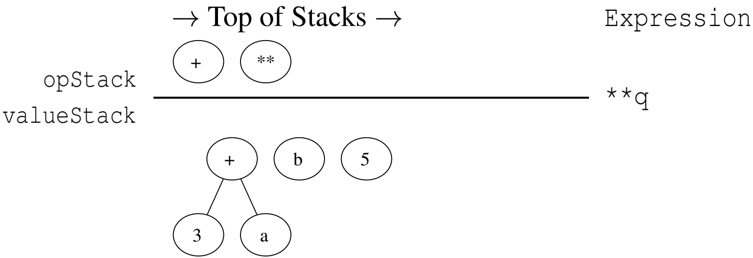


`5**q`

Push operation ** onto `opStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

Expression

opStack

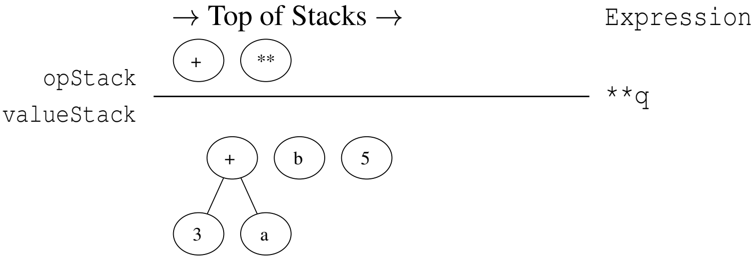
valueStack

`**q`

Push value `5` onto `valStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

`Expression`

`opStack`

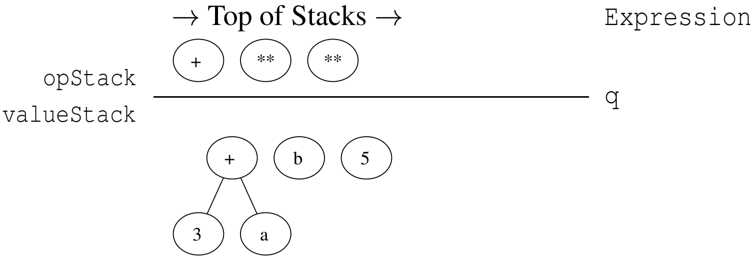
`valueStack`

`**q`

`Expression`'s **right-associative** `**` precedence is ≥ `**` (`opStack`) precedence

# The Shunting Yard Algorithm (Ex 1)

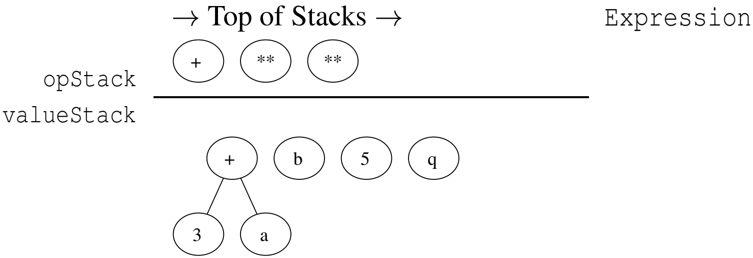
`3 + a + b ** 5 ** q`



Push operation ** onto `opStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

`Expression`

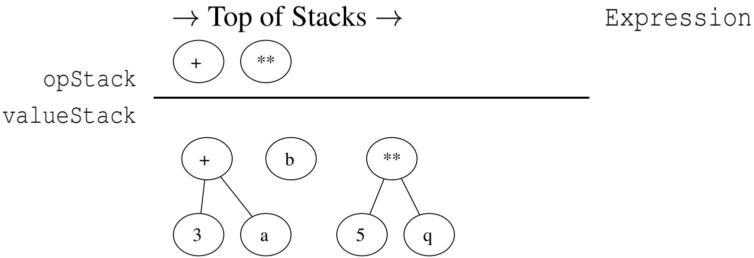
`opStack`

`valueStack`

Push value `q` onto `valStack`

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



→ Top of Stacks →

Expression

opStack

valueStack

The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

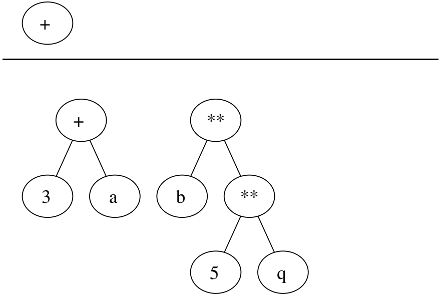
# The Shunting Yard Algorithm (Ex 1)

```
3 + a + b ** 5 ** q
```

→ Top of Stacks → `Expression`

`opStack`

`valueStack`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

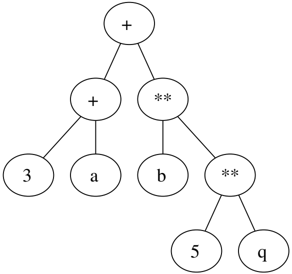
# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`

→ Top of Stacks → `Expression`

`opStack`

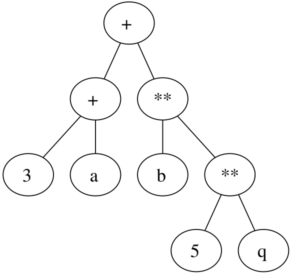
`valueStack`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

# The Shunting Yard Algorithm (Ex 1)

`3 + a + b ** 5 ** q`



← Expression Tree!

Notice how right-associative exponentiation must be calculated **first** — results of higher precedence operations are the **children** of lower precedence ops.

# The Shunting Yard Algorithm (Ex 2)

```
-x * a / ( b + 5 ) ** q
```

Try drawing the expression tree first . . .

can you predict the algorithm results?

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`

We begin with an empty `valueStack` and `opStack`, and traverse the elements of an expression from left to right.

→ Top of Stacks → `Expression`

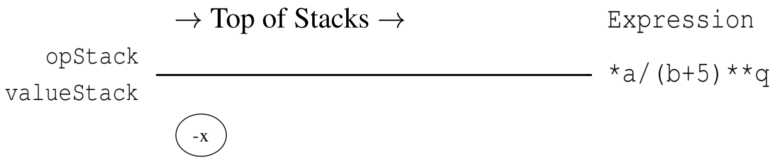
`opStack`

`valueStack` `-x*a/(b+5)**q`

Begin. . .

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

`Expression`

`opStack`

`valueStack`

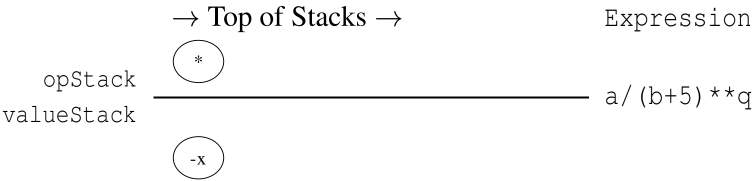
`*a/(b+5)**q`

-x

Push value `-x` onto `valStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

valueStack

*

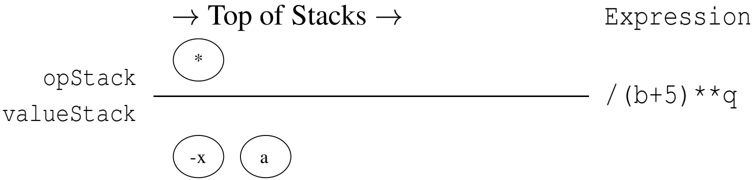
-x

a/(b+5)**q

Push operation * onto `opStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

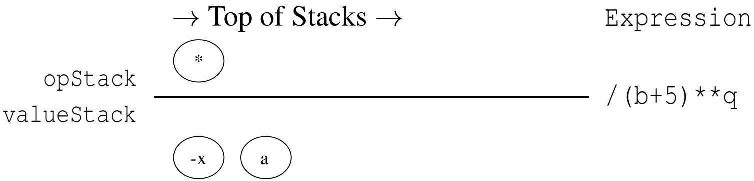
opStack

valueStack

Push value `a` onto `valStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

*

/(b+5)**q

valueStack

-x

a

Expression's **left-associative** / precedence is
≤ * (opStack) precedence → pop the opStack

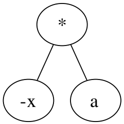
# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`

→ Top of Stacks → `Expression`

`opStack` `/(b+5)**q`

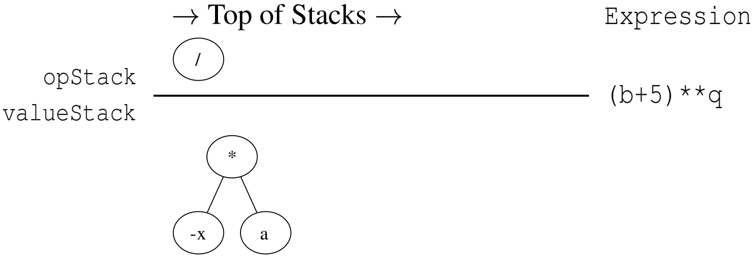
`valueStack`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

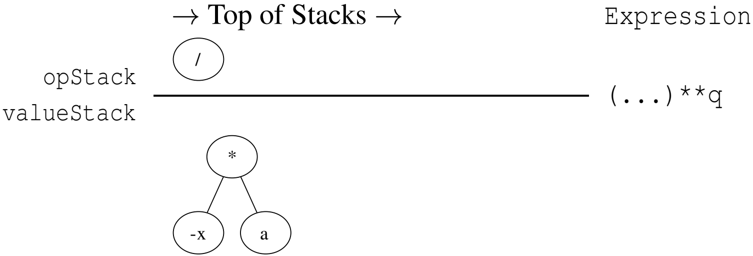
valueStack

`(b+5)**q`

Push operation / onto `opStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

valueStack

(...)**q

Recursive call for parenthetical grouping

# The Shunting Yard Algorithm (Ex 2)

```
-x * a / ( b + 5 ) ** q
```

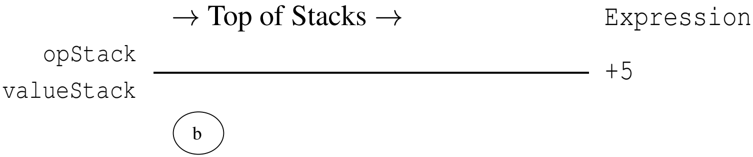
→ Top of Stacks →

```
                                              Expression
   opStack
valueStack                                    b+5
```

Begin the recursive call for subexpression tree

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

`Expression`

`opStack`

`+5`

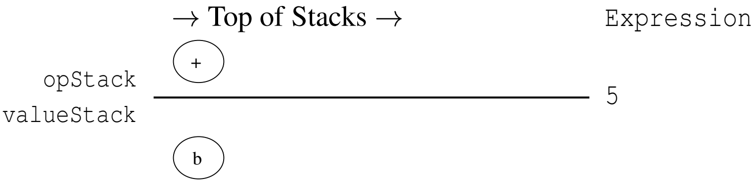
`valueStack`

b

Push value `b` onto `valStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

+

valueStack

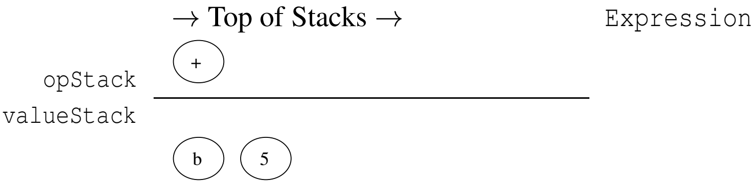
b

5

Push operation + onto `opStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



Push value `5` onto `valStack`

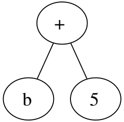
# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`

→ Top of Stacks → `Expression`

`opStack`

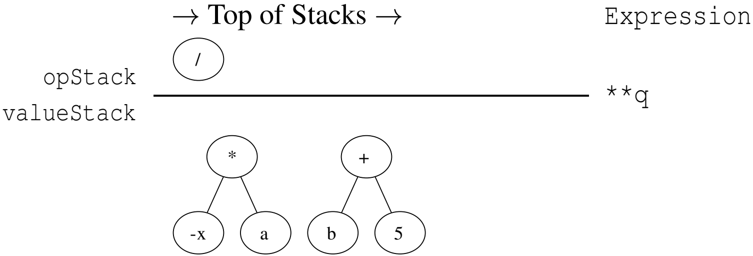
`valueStack`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

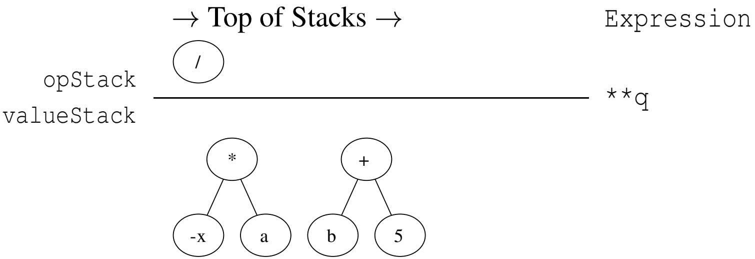
valueStack

`**q`

Recursive call returns — place subexpression tree on `valStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

`Expression`

`opStack`

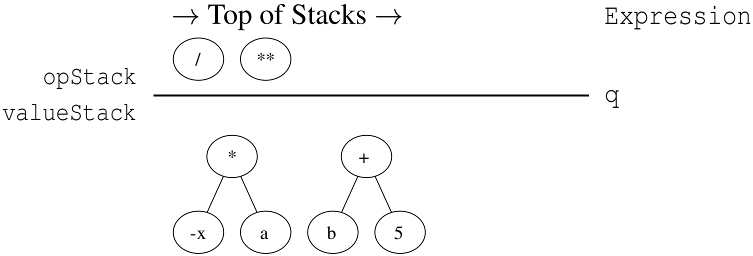
`valueStack`

`**q`

`Expression`’s **right-associative** `**` precedence is ≥ / (`opStack`) precedence

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

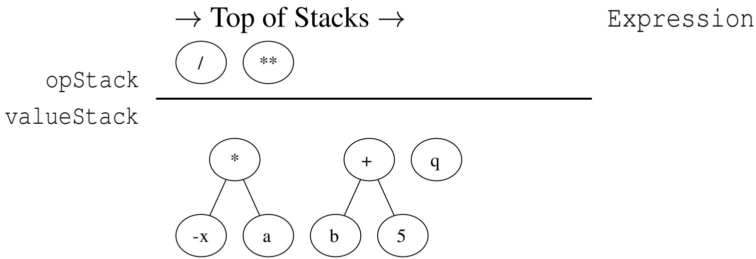
valueStack

q

Push operation ** onto `opStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

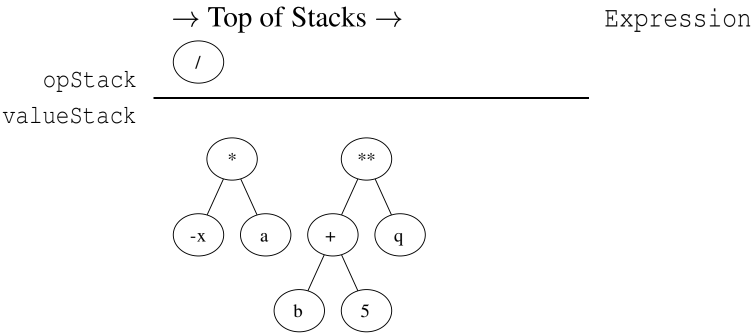
opStack

valueStack

Push value `q` onto `valStack`

# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`



→ Top of Stacks →

Expression

opStack

valueStack

The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

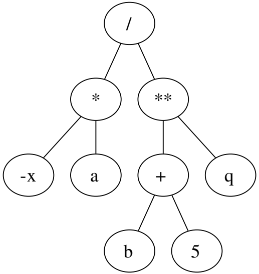
# The Shunting Yard Algorithm (Ex 2)

`-x * a / ( b + 5 ) ** q`

→ Top of Stacks → `Expression`

`opStack`

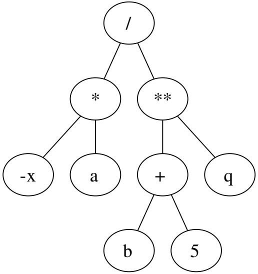
`valueStack`



The top operation on the `opStack` binds with the top two elements of the `valueStack` the result is a new value that is pushed onto the `valueStack`.

# The Shunting Yard Algorithm (Ex 1)

```
-x * a / ( b + 5 ) ** q
```

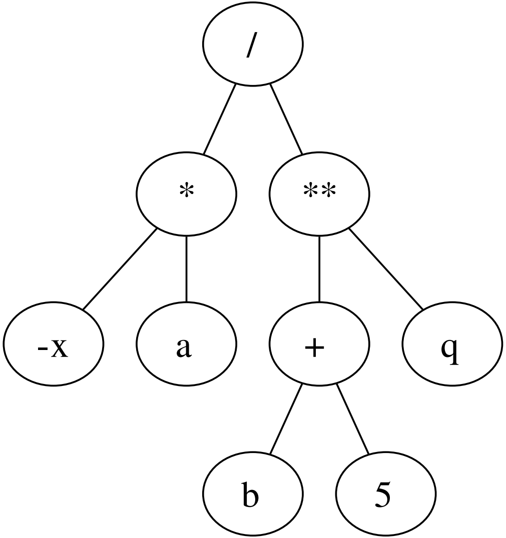


← Expression Tree!

Notice how we handled parenthetical grouping with recursion — `(...)` results are always "low" on the tree.

# The Next Step . . .

```
-x * a / ( b + 5 ) ** q
```



Now that we can generate **expression trees**, how do we take advantage of the tree structure to methodically generate **machine code**?