Arithmetic Properties through Grammars

There are specialized algorithms[1] for parsing arithmetic expressions (written programmatically) such as

$$a + bc - 4^x \quad \equiv \quad \texttt{a + b*c - 4**x}$$

But important arithmetic properties such as **order of operations** and **associativity** can be easily expressed using **context free grammars**. All non-trivial languages embed the rules of mathematics into their language definition.[2]

[1] `show_shunting.pdf`

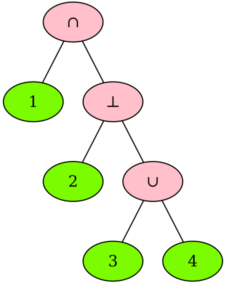
[2] Would you want to work in a language where `11+3 * 2` is `28`?

Arithmetic Properties through Grammars

We want to avoid our "built-in" knowledge of arithmetic, so we have to focus on the mechanics of this technique. So we'll use $\cup$, $\perp$ and $\cap$ to represent our operators.

operator	precedence	associativity
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

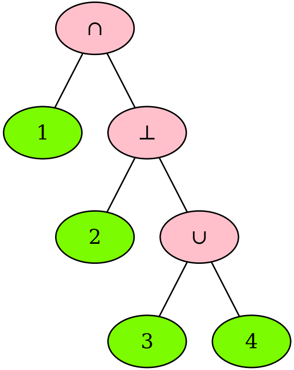
Input: $1 \cap 2 \perp 3 \cup 4$



Grammar Patterns for Arithmetic Operations

Input: $1 \cap 2 \perp 3 \cup 4$

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$



op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Notice how there is a **nesting** or **layering** of the grammar non-terminals from **lowest to highest precedence operators** (think of numbers as the highest precedence terms)

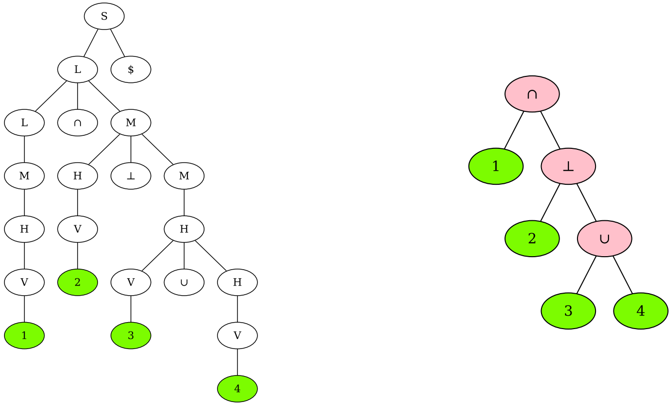
$$L \rightarrow M \rightarrow H \rightarrow V$$

Grammar Patterns for Arithmetic Operations

Input: $1 \cap 2 \perp 3 \cup 4$

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?



We'll use **syntax directed translation** in the course to perform the simplification from "raw parse tree" to more concise **expression trees** *as parsing takes place*.

For now we want the larger raw parse tree to see the grammar mechanics in action.

Grammar Patterns for Arithmetic Operations

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

RECALL! $H \Rightarrow^{+} \delta V \pi$ means the grammar symbol V can be **derived from** non-terminal H with one or more substitutions.

Both rules 6 and 7 of this expression grammar satisfy the $H \Rightarrow^{+} \delta V \pi$ assertion, since δ and π are $(N \cup \Sigma)^{*}$.

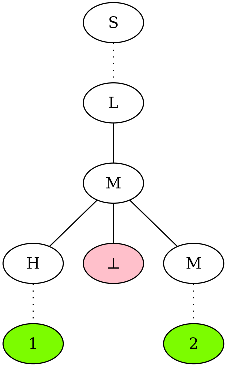
But only rule 7 satisfies $H \Rightarrow^{+} V$, since the absence of δ and π means we can't match (or generate) any other language terminals from Σ into the derivation.

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



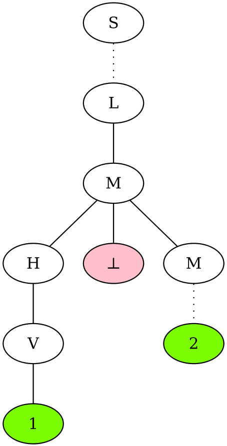
The $1 \perp$ is "captured" by rule 4, because rule 4 is the only way to incorporate the $\perp$ operator into our parse.

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



We need $H \Rightarrow^{+} num$ to successfully parse the initial 1. That's not too difficult:

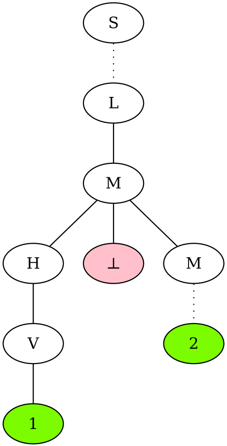
$$H \rightarrow V \rightarrow num$$

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \to L\,\$$
2	$L \to L \cap M$
3	$L \to M$
4	$M \to H \perp M$
5	$M \to H$
6	$H \to V \cup H$
7	$H \to V$
8	$V \to num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



We need $H \Rightarrow^{+} num$ to successfully parse the initial 1. That's not too difficult:

$$H \to V \to num$$

Now we need $M \Rightarrow^{+} \delta\, num\, \pi$ in order to parse the the 2 from the input. We could use several sequences of substitutions to accomplish this:

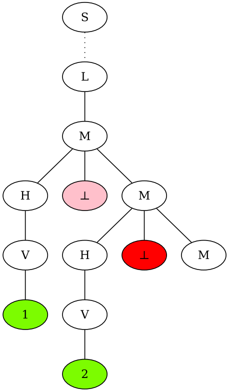
1. Rules 4, 7, 8
2. Rules 5, 6, 8

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



Now we need $M \Rightarrow^{+} \delta\, num\, \pi$ in order to parse the the 2 from the input. We could use several sequences of substitutions to accomplish this:

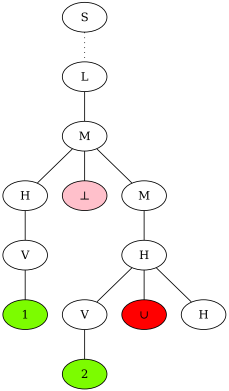
1. Rules 4, 7, 8
   This would require a $\perp$ after the 2 in the input (there isn't one).
2. Rules 5, 6, 8

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



Now we need $M \Rightarrow^{+} \delta\, num\, \pi$ in order to parse the the 2 from the input. We could use several sequences of substitutions to accomplish this:

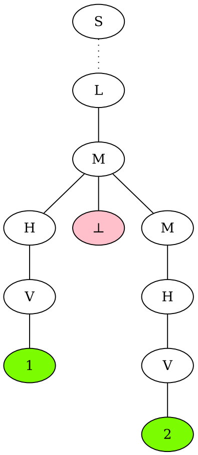
1. Rules 4, 7, 8
2. Rules 5, 6, 8
   This would require an $\cup$ after the 2 in the input (there isn't one).

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



The only way to accomplish $M \Rightarrow^{+} \delta\, num\, \pi$ for the 2 in the input is using rules 5, 7 and 8 which don't introduce terminals **absent from the input**.

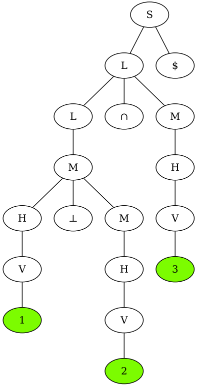
The remaining input, $\cap\ 3$, **must be incorporated into the parse tree between the root S and its child L tree** created to parse the $\perp$ operator.

Higher Precedence Followed by Lower Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \perp 2 \cap 3$



With this insight, it's not difficult to deduce the remaining portion of the tree. We need rule 2 (it's the only rule with the $\cap$ operator) and conveniently we can incorporate it by the recursive property of L in rule 2.

And this is how higher-to-lower precedence can be evaluated in an **unambiguous** manner when the operator grammar is written correctly.

Let's convice ourselves this works for lower-to-higher precedence. . .

Lower Precedence Followed by Higher Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

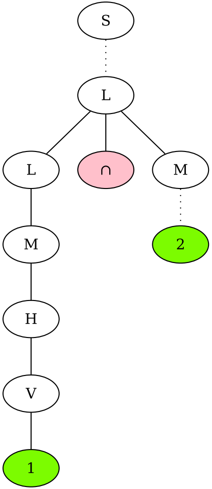
op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

The $1\ \cap$ is “captured” by rule 2 (it’s the only rule with the $\cap$ operator in its RHS).

We need $L \Rightarrow^+ num$ to successfully parse the initial 1. That’s not too difficult:

$$L \rightarrow M \rightarrow V \rightarrow num$$

Input: $1 \cap 2 \perp 3$



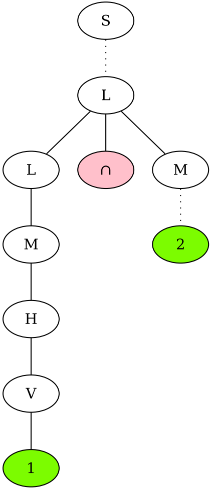
Lower Precedence Followed by Higher Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Now we need $M \Rightarrow^{+} \delta\, num\, \pi$.

Input: $1 \cap 2 \perp 3$



Lower Precedence Followed by Higher Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

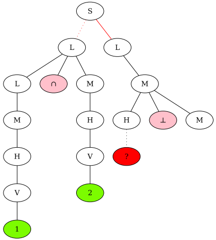
We could use rules 5, 7 and 8; but then we must parse the $\perp$ following 2 in the input.

Could we use rules 3 and 4 for the $\perp$?

No:

1. There is no $S \rightarrow L\,L$ production rule
2. and rule 4's RHS H would have to $H \Rightarrow^{+} \delta\, num\, \pi$ but there isn't another num between 2 and $\perp$.

Input: $1 \cap 2 \perp 3$



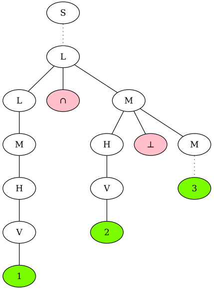
Lower Precedence Followed by Higher Precedence

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

The only way to parse the $2\ \perp$ phrase of the input is to resolve $M \Rightarrow^{+} \delta\, num\ \perp\ \pi$ using rule 4.

Input: $1\ \cap\ 2\ \perp\ 3$



Lower Precedence Followed by Higher Precedence

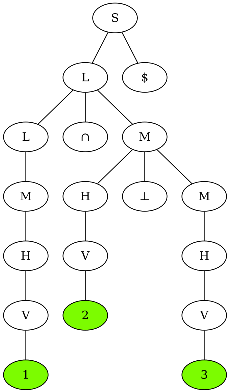
#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

The only way to parse the $2\ \perp$ phrase of the input is to resolve $M \Rightarrow^{+} \delta\, num \perp \pi$ using rule 4.

And rules 5, 7 and 8 allow $M \Rightarrow^{+} num$.

Input: $1 \cap 2 \perp 3$

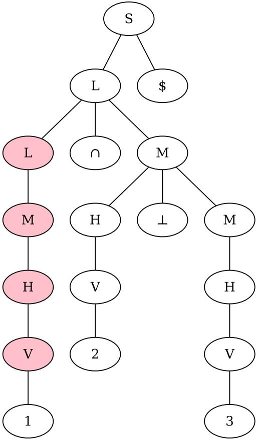


Grammar Pattern for Operator Precedence — The Takeaway

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

Input: $1 \cap 2 \perp 3$

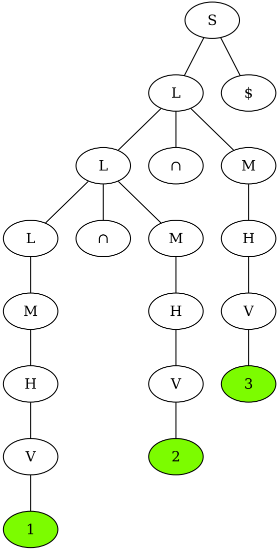


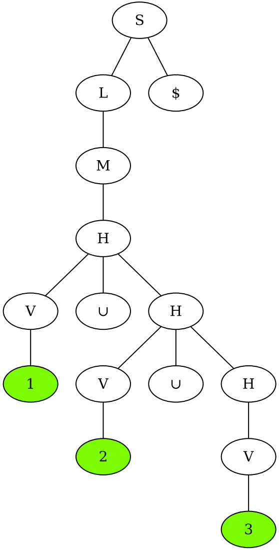
Operator precedence is encoded into a grammar as strict "levels" or "layers". A higher precedence level **must be rewritten** as the RHS non-terminal of the next lower precedence level in order for its value to "flow up" the expression tree into the final result.

Grammar Pattern for Operator Associativity

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	?
$\perp$	middle (M)	?
$\cup$	highest (H)	?

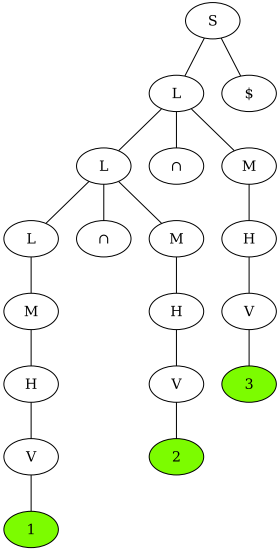




Left Associative Op: $1 \cap 2 \cap 3$

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	**left**
$\perp$	middle (M)	?
$\cup$	highest (H)	?



There are two $\cap$ operators in our input source, so we'll have to use two applications of rule 2 (the only rule permitting the $\cap$ symbol).

The grammar does not permit L to be below Ms in the parse tree. Why? There are no production rules $M \Rightarrow^{+} \delta\, L\, \pi$. So there is only one way to combine these two rewrite rules — recursing down the LH operand:

$$\begin{array}{lll} S & \Rightarrow & L\ \$ \\ S & \Rightarrow & L\ \cap_2\ M_2\ \$ \\ S & \Rightarrow_{lm} & L_1\ \cap_1\ M_1\ \cap_2\ M_2 \\ S & \Rightarrow^{+} & 1\ \cap_1\ 2\ \cap_2\ 3 \end{array}$$

This left recursion bound the 2 in the input to $\cap_1$, which is **left associative**.

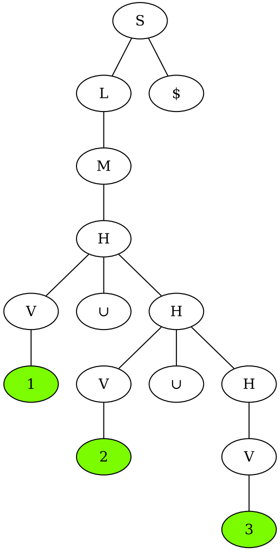
Right Associative Op: $1 \cup 2 \cup 3$

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$

op	prec	assoc
$\cap$	lowest (L)	**left**
$\perp$	middle (M)	**right**
$\cup$	highest (H)	**right**

Likewise, the **right recursive rules** 4 and 6 means means $\perp$ and $\cup$ are **right associative** operators.

$$
\begin{array}{lll}
S & \Rightarrow^{+} & H\ \$ \\
S & \Rightarrow & V_1\ \cup_1\ H\ \$ \\
S & \Rightarrow^{+}_{rm} & V_1\ \cup_1\ V_2\ \cup_2\ H \\
S & \Rightarrow^{+} & 1\ \cup_1\ 2\ \cup_2\ 3
\end{array}
$$



Grammar Pattern for Parenthetical Precedence (Rule 9)

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$
9	$V \rightarrow (\, L \,)$

op	**prec**	**assoc**
$\cap$	lowest (L)	left
$\perp$	middle (M)	right
$\cup$	high (H)	right
$(\cdot)$	highest (V)	none

Parenthetical overrides of precedence are straightforward to express in a grammar.

1. Identify the **non-terminal** that captures the lowest precedence expressions (L in this grammar).
2. Identify the **non-terminal** that captures values, variables, numbers, and literals (in this grammar, non-terminal V).
3. Modify the grammar to treat $(\, L \,)$ as if it were a variable or value (rule 9); where $($ and $)$ are the open and closing bracket symbols used for grouping in the language.

Grammar Pattern for Parenthetical Precedence (Rule 9)

Precedence: $(\ 1 \cap 2\ ) \cup 3$ Associativity: $(\ 1 \perp 2\ ) \perp 3$

#	Rules
1	$S \rightarrow L\,\$$
2	$L \rightarrow L \cap M$
3	$L \rightarrow M$
4	$M \rightarrow H \perp M$
5	$M \rightarrow H$
6	$H \rightarrow V \cup H$
7	$H \rightarrow V$
8	$V \rightarrow num$
9	$V \rightarrow (\ L\ )$

op	prec	assoc
$\cap$	lowest (L)	left
$\perp$	middle (M)	right
$\cup$	high (H)	right
$(\cdot)$	highest (V)	none